Abstract

Recent work of Biedermann and Röndigs has translated Goodwillie’s calculus of functors into the language of model categories. Their work focuses on symmetric multilinear functors and the derivative appears only briefly. In this paper we focus on understanding the derivative as a right Quillen functor to a new model category. This is directly analogous to the behaviour of Weiss’s derivative in orthogonal calculus and gives a better idea of the relation between the two forms of calculus. The immediate advantage of this new category is that we obtain a streamlined and more informative proof that the \( n \)-homogeneous functors are classified by spectra with a \( \Sigma_n \)-action.
1 Introduction

1.1 Overview

We will construct the following diagram of Quillen functors (with left adjoints on top or on the left) and show that each adjunction is a Quillen equivalence. The top line of this diagram provides a lifting of the homotopy classification of $n$-homogeneous functors from \cite{Goo03} to model categories.

![Diagram of Quillen adjunctions](image1)

Figure 1: Diagram of Quillen adjunctions

This lifting of the classification gives a shorter and more direct classification than that of \cite{BR13} (see Figure 2 for their corresponding diagram). This description puts Goodwillie calculus into the same format as orthogonal calculus, a result which we plan to develop upon in a sequel to this paper. In particular, the category $\Sigma_n \ltimes (\mathcal{W}_n \text{Top})$ is very similar in its construction to the intermediate category $O(n)\mathcal{E}_n$ of \cite[Section 3]{BO13}, which in our notation would be written as $\mathcal{J}_n \ltimes (O(n)\text{Top})$.

In both \cite{BR13} and \cite{Goo03} there is no direct comparison nor Quillen equivalence between spectra and $n$-homogeneous endofunctors of spaces. Instead they pass through several other categories as shown below in Figure 2. In contrast, we show that the derivative construction (denoted $\text{diff}_n$, see Definition 3.7) naturally takes values in the new category $\Sigma_n \ltimes (\mathcal{W}_n \text{Top})$. Then we establish that this construction is a Quillen equivalence. Furthermore we construct a Quillen equivalence between $\Sigma_n \ltimes (\mathcal{W}_n \text{Top})$ and spectra with a $\Sigma_n$-action (denoted $\Sigma_n \otimes \mathcal{W}Sp$). This involves one less adjunction and fewer categories. Hence from the model category point of view, the classification of $n$-homogeneous functors does not require symmetric multilinear functors. This is the same pattern as in \cite{BO13}.

The category $\Sigma_n \ltimes (\mathcal{W}_n \text{Top})$ is a relatively standard construction of equivariant spectra, similar to the constructions of equivariant orthogonal spectra of \cite{MM02}. Thus if we are prepared

![Diagram of Biedermann-Röndigs \cite{BR13}](image2)

Figure 2: Diagram of Biedermann-Röndigs \cite{BR13}
to work with this category rather than spectra with a $\Sigma_n$-action we will have a one-stage classification of homogeneous functors in terms of spectra. We also claim that our category $\Sigma_n \ltimes (W_n \text{Top})$ is no more complicated than the category of symmetric functors. See Section 4 for a definition of $\Sigma_n \ltimes (W_n \text{Top})$ and Section 7 for a comparison with symmetric multi-linear functors.

Another useful aspect of this work is that we choose to work with the category $W_{\text{Top}}$ as our model of homotopy functors. Every object of this category is a homotopy functor, which removes the need for the homotopy functor model structure, a prominent feature of [BCR07, BR13]. Additionally, there is a strong body of work on $W_{\text{Top}}$ and related spectra, see [MMSS01].

In a sequel to this paper we intend to give a formal comparison between the model categories used for Goodwillie calculus and the model categories used for orthogonal calculus ([BO13]), verifying what are currently folk results relating $n$-excisive functors and $n$-polynomial functors.

1.2 Recent developments

Goodwillie calculus of homotopy functors is a highly successful method of studying functors with source and target either spaces or spectra. The original development of this method is given in the three papers by Goodwillie: [Goo03], [Goo92] and [Goo90]. It has seen a number of applications, including Waldhausen’s algebraic $K$-theory of spaces, Snaith-type splittings (by Arone [Aro99]) and classical homotopy theory (e.g. [Beh12] and [AM99]).

Let $F$ be a functor from the category spaces to itself which preserves weak homotopy equivalences; such functors are called homotopy functors. The full generality of Goodwillie’s calculus allows for working over a fixed space (or spectrum) $Y$. In this paper we will focus on the fundamental case where $Y$ is just the point, denoted $\ast$. Examples of homotopy functors include $\Omega^\infty \Sigma^\infty$, $- \wedge X$ and $\text{Hom}(X, -)$ for $X$ a CW-complex, any topologically enriched functor, and Waldhausen’s algebraic $K$-theory. Goodwillie’s calculus associates to $F$ a tower of approximations

$$
\begin{array}{cccc}
P_3 F & \rightarrow & D_3 F \\
P_2 F & \rightarrow & D_2 F \\
P_1 F & \rightarrow & D_1 F \\
P_0 F & \rightarrow & F
\end{array}
$$

The functor $P_n F$ is the closest (meaning universal in the homotopy category) functor to $F$ that is $n$–excisive: it takes strongly homotopy cocartesian $(n + 1)$–cubical diagrams to homotopy cartesian diagrams. For $n = 1$, this means that $P_1 F$ takes homotopy pushout squares to homotopy pullback squares. For $n = 0$ this means that the functor $P_0 F$ is constant — it is objectwise weakly equivalent to $F$ of the final space $Y$, which for us, is the point. The functor $D_0 F$ is the closest $n$–homogeneous functor to $F$. That is, it is the closest $n$–excisive functor $G$ such that $P_{n-1} G$ is objectwise weakly equivalent to a point.

A major result of Goodwillie is the classification of finitary (roughly, determined by its values on finite CW complexes) $n$–homogeneous functors [Goo03, Section 3] up to homotopy. That is, for any $F$ and any $n \geq 1$ there is a spectrum $\partial_n F$ with $\Sigma_n$-action, that is unique up to
homotopy, such that the \( n \)-homogeneous approximation to \( F \) is as follows.

\[
D_n F(X) \simeq \Omega^\infty ((\partial_n F(\ast) \wedge X^\wedge n)_{h\Sigma_n})
\]  

(1)

Given the spectra \( \partial_n F(\ast) \) and the space \( X \) one can attempt to calculate the sequence of approximations and hence gain understanding of the behaviour of \( F \). As the linearisation of the identity of spaces is \( \Omega^\infty \Sigma^\infty \), i.e. the “stable homotopy” of whatever space one inputs, one thinks of the Goodwillie tower as interpolating between unstable and stable phenomena.

Recently, Biedermann, Chorny and Röndigs [BCR07] and Biedermann and Röndigs [BR13] have completed Goodwillie’s recommendation (in [Goo03, p.655]) that the functor calculus should be set up in the language of model categories. In particular, the latter of those two papers takes Goodwillie’s classification of homogeneous functors and shows how it lifts from an equivalence of homotopy categories to the much more structured and useful notion of a Quillen equivalence. They closely follow the same pattern as Goodwillie’s paper [Goo03], which involves several intermediate categories, as seen in Figure 2.

They (i.e. [BCR07, BR13, Goo03]) start with the category of \( n \)-fold symmetric functors. Such a functor \( H \) takes as input \( n \) spaces and gives as output a space \( H(X_1, \ldots, X_n) \), such that a change in the ordering \( \sigma \in \Sigma_n \) induces an isomorphism

\[
H(X_1, \ldots, X_n) \xrightarrow{\cong} H(X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(n)})
\]

Furthermore, \( H \) preserves weak homotopy equivalences in all of its variables. Given any functor \( F \) one can construct an \( n \)-fold symmetric functor called the cross-effect of \( F \), denoted \( \text{cr}_n F \).

To relate this to the tower of approximations above, there is a multivariable version of \( P_1 \), called \( P_{1, \ldots, 1} \) which satisfies the relation:

\[
\text{cr}_n P_n F \simeq P_{1, \ldots, 1} \text{cr}_n F
\]

A functor \( H \) which is objectwise weakly equivalent to \( P_{1, \ldots, 1} H \) is called an \( n \)-fold symmetric multilinear functor.

With these notions clear, we can explain the categories appearing in the main diagram of Biedermann and Röndigs, Figure 2. They use based simplicial sets as their category of “spaces”, denoted \( \mathcal{S} \). The full subcategory of \( \mathcal{S} \) on the class of finite simplicial sets is denoted \( \mathcal{S}^f \). We let \( \text{Sp} \) denote the category of Bousfield Friedlander spectra in simplicial sets and \( \Sigma_n \circ \text{Sp} \) the category of spectra with an action of \( \Sigma_n \). The subscript \( n \)-homog indicates that we have chosen the model structure on functors from \( \mathcal{S} \) to \( \text{Sp} \) (or \( \text{Sp}^f \)) so that the homotopy category is the category of \( n \)-homogenous functors. Similarly, a subscript \( n \)-excisive indicates that we have chosen the model structure on symmetric functors with target \( \mathcal{S} \) (or \( \text{Sp} \)) such that the homotopy category is the category of symmetric multilinear functors. The functors in this diagram are all the right adjoints of Quillen equivalences. For an explanation of these categories and a comparison (indeed, a Quillen equivalence) with our own, see Section 7.

1.3 Organisation

In Section 2 we remind the reader of some important model category definitions and introduce \( \text{WTop} \), the category of functors that we will use to model homotopy functors. We then follow the structure of [BR13] and establish model structures on \( \text{WTop} \) analogous to their work. Specifically, in Section 3 we define the cross effect model structure, the \( n \)-excisive model structure and the \( n \)-homogeneous model structure. Since we have chosen to work in \( \text{WTop} \), we avoid
having to use a homotopy functor model structure as in [BR13], but must in recompense pay
attention to certain smallness conditions. When developing the homogeneous model structure
we focus on the cofibrant generation and leave the use of [Bou01, Theorem 9.3] until last, the
opposite approach to [BR13, Section 5].

With these basics completed, we can turn to the construction of the new category \( \Sigma_n \ltimes (W_n \text{Top}) \).
In Section 4 we start by giving the construction of the stable model structure on spectra
with a \( \Sigma_n \)–action. This serves as a warm-up for constructing the stable model structure on
\( \Sigma_n \ltimes (W_n \text{Top}) \). Section 5 establishes the Quillen equivalence between \( \Sigma_n \ltimes (W_n \text{Top}) \) and spectra
with a \( \Sigma_n \)–action. The Quillen equivalence between \( n \)-homogeneous functors and \( \Sigma_n \ltimes (W_n \text{Top}) \)
induced by differentiation is established in Section 6. This is the primary result of this part of
the paper. We finish by giving the Quillen equivalence between symmetric multilinear functors
and \( \Sigma_n \ltimes (W_n \text{Top}) \) in Section 7.

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2 Model structures on spaces and functors

2.1 Model category background

We start by reviewing a small amount of model category concepts that we will use throughout
the paper. The conditions we use are essentially those which make arbitrary model categories
most like spaces. Firstly the ability to pushout or pullback weak equivalences (properness).
Secondly that it has a good notion of cellular approximation (cofibrantly generated). Similarly
to [MMSS01], we use topological, rather than simplicial model categories, and provide a short
definition of those.

We first point out when we say a category is topological we mean that it is enriched in \( \text{Top} \)
in the sense of [Kel05] (the category has spaces of morphisms and continuous composition).
This concept is independent of the model structure. Whereas a model category is said to be
topological if it satisfies the following definition, which is analogous to the concept of a simplicial
model category.

**Definition 2.1 [MMSS01, Definition 5.12]** For maps \( i : A \rightarrow X \) and \( p : E \rightarrow B \) in a model
category \( \mathcal{M} \), let the map below be the map of spaces induced by \( \mathcal{M}(i, id) \) and \( \mathcal{M}(i, p) \) after passing
to the pullback.

\[
\mathcal{M}(i^*, p_*) : \mathcal{M}(X, E) \rightarrow \mathcal{M}(A, E) \times_{\mathcal{M}(A,B)} \mathcal{M}(X, B)
\]

A model category \( \mathcal{M} \) is **topological**, provided that \( \mathcal{M}(i^*, p_*) \) is a Serre fibration if \( i \) is a
\( q \)–cofibration and \( p \) is a \( q \)–fibration and is a weak equivalence if, in addition, either \( i \) or \( p \)
is a weak equivalence.

**Definition 2.2 [Hir03, Definition 11.1.1]** Let \( \mathcal{M} \) be a model category, and let the following be
a commutative square in $\mathcal{M}$:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\ i} & & \downarrow{\ j} \\
C & \xrightarrow{g} & D 
\end{array}
$$

$\mathcal{M}$ is called **left proper** if, whenever $f$ is a weak equivalence, $i$ a cofibration, and the square is a pushout, then $g$ is also a weak equivalence. $\mathcal{M}$ is called **right proper** if, whenever $g$ is a weak equivalence, $j$ a fibration, and the square is a pullback, then $f$ is also a weak equivalence. $\mathcal{M}$ is called **proper** if it is both left and right proper.

This concept can also be phrased as the set of (co)fibrations being closed under (co)base change.

**Definition 2.3** [Hir03, Definition 13.2.1] A **cofibrantly generated model category** is a model category $\mathcal{M}$ with sets of maps $I$ and $J$ such that $I$ and $J$ support the small object argument (see [MP12, Definitions 15.1.1 and 15.1.7.]) and

1. a map is a trivial fibration if and only if it has the right lifting property with respect to every element of $I$, and

2. a map is a fibration if and only if it has the right lifting property with respect to every element of $J$.

### 2.2 Three model structures on spaces

We let $\mathsf{Top}$ denote the category of based compactly generated based spaces, also known as the category of weak Hausdorff $k$-spaces. This category is a closed symmetric monoidal topological category, with monoidal product the smash product $- \wedge -$ and unit $S^0$. The internal function object is denoted $\mathsf{Top}(-, -)$.

There are three model structures that we use on $\mathsf{Top}$, the $q$- (“Quillen”) model structure, the $h$- (“Hurewicz) model structure and the $m$- (“Mixed”)model structure (see Cole [Col06]). In a general model category, $q$ tends to mean the (co)fibrations and weak equivalences are given by the specified “standard” model structure, and the $h$-(co)fibrations and weak equivalences are defined using diagrams analogous to the Homotopy Extension Lifting Property of spaces.

The following definitions and constructions are all given for $\mathsf{Top}$. We follow the treatment in [MP12, Part 4 Section 17-18]. Note that the $h$-cofibrations defined below are closed inclusions of spaces, see [MMSS01, Page 457].

**Theorem 2.4** [MP12, Theorem 17.1.1, Corollary 17.1.2] The category $\mathsf{Top}$ of based spaces has a monoidal and proper model structure, the $h$-**model structure**, where the

- weak equivalences are the homotopy equivalences
- fibrations are the Hurewicz fibrations
- cofibrations are the $h$-cofibrations (defined by the left lifting property).

All spaces are both fibrant and cofibrant.

Let $I_{\mathsf{Top}}$ be the set of inclusions $S^{n-1} \rightarrow D^n_+$, $n \geq 0$ and $J_{\mathsf{Top}}$ the set of maps $i_0 : D^n_+ \rightarrow (D^n \times I)_+$, $n \geq 0$. 

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Theorem 2.5 [MP12, Theorem 17.2.2, Corollary 17.2.4] The category \( \text{Top} \) of based spaces has a cofibrantly generated monoidal and proper model structure, the \( q \)-\textbf{model structure}, where the

- weak equivalences are the weak homotopy equivalences
- fibrations are the Serre fibrations (those maps that satisfy the right lifting property with respect to \( I \)).
- cofibrations are the \( q \)-cofibrations (defined by the left lifting property).

All spaces are fibrant. The generating sets for this model category are the sets \( I_{\text{Top}} \) and \( J_{\text{Top}} \) given immediately above.

To construct a mixed model structure, one starts with two model structures \((W_h, C_h, F_h)\) and \((W_q, C_q, F_q)\) which satisfy \( W_h \subset W_q \), \( F_h \subset F_q \) and \( C_q \subset C_h \). The makes a new model structure with \( W_q \) the weak equivalences and \( F_h \) the fibrations. For us, these are the \( h \) and \( q \) model structures on \( \text{Top} \). This then allows us to construct a “mixed” model structure as follows.

Theorem 2.6 [MP12, Theorem 17.4.2, Corollary 17.4.3] The category \( \text{Top} \) of based spaces has a monoidal and proper model structure, the \( m \)-\textbf{model structure}, where the

- weak equivalences are the weak homotopy equivalences
- fibrations are the Hurewicz fibrations
- cofibrations defined by the left lifting property with respect to Hurewicz fibrations which are also \( q \)-equivalences.

2.3 The category \( \mathcal{W}\text{Top} \) of topological functors

Goodwillie calculus studies equivalence-preserving functors from the category of based spaces to itself. We therefore need a good model for the category of such functors. In this section we introduce \( \mathcal{W}\text{Top} \) and show how it is a good model for the category of homotopy functors.

Let \( \mathcal{W} \) be the category of based spaces homeomorphic to finite CW complexes. We note immediately that \( \mathcal{W} \) is Top-enriched, but not \( \mathcal{W} \)-enriched. We define \( \mathcal{W}\text{Top} \) to be the category of \( \mathcal{W} \)-\textbf{spaces}: continuous functors from \( \mathcal{W} \) to \( \text{Top} \) (for full details see [MMSS01]). In particular, an \( X \in \mathcal{W}\text{Top} \) consists of the following information: a collection of based spaces \( X(A) \) for each \( A \in \mathcal{W} \) and a collection of maps of based spaces

\[
X_{A,B} : \mathcal{W}(A, B) \to \text{Top}(X(A), X(B))
\]

for each pair \( A, B \) in \( \mathcal{W} \). These maps must be compatible with composition and also associative and unital. The map \( X_{A,B} \) induces a structure map:

\[
X(A) \land \mathcal{W}(A, B) \to X(B)
\]

The category \( \mathcal{W}\text{Top} \) is complete and cocomplete with limits and colimits taken objectwise. This category is tensored and cotensored over based spaces. For a functor \( X \) in \( \mathcal{W}\text{Top} \) and a based space \( A \), the tensor \( X \land A \) is the levelwise smash product. The cotensor \( \text{Top}(A, X) \) is the levelwise function space. The category \( \mathcal{W}\text{Top} \) is also enriched over based spaces, with the space of natural transformations from \( X \) to \( Y \) given by

\[
\text{Nat}(X, Y) = \int_{A \in \mathcal{W}} \text{Top}(X(A), Y(A))
\]
The category $\text{WTop}$ is a closed symmetric monoidal category by \cite[Theorem 1.7]{MMSS01}. The smash product and internal function object are defined as follows, where $X$ and $Y$ are objects of $\text{WTop}$ and $A \in W$.

$$(X \wedge Y)(A) = \int_{B,C \in W}^{B,C \in \text{Top}} X(B) \wedge Y(C) \wedge W(B \wedge C, A)$$

$$\text{Hom}(X, Y)(A) = \int_{B \in W} \text{Top}(X(B), Y(A \wedge B))$$

We may also describe $\text{Hom}(X, Y)(A)$ as $\text{Nat}(X, Y(A \wedge -))$ and we note that these formulas agree with the standard formalism of \cite{Day70}. There is another important natural construction that we will make use of. Let $X$ be an object of $\text{WTop}$, then the **assembly map** of $X$ is

$$a_{A,B} : X(A) \wedge B \to X(A \wedge B)$$

It may be defined as the following composition, where the final map is the structure map of $X$.

$$X(A) \wedge B \cong X(A) \wedge W(S^0, B) \xrightarrow{\text{Id}_{X(A \wedge -)}} X(A) \wedge W(A, A \wedge B) \xrightarrow{X} X(A \wedge B)$$

The existence of the assembly map tells us that $X$ takes homotopic maps to homotopic maps (compose the assembly map with $X$ applied to the homotopy between the maps). Since $W$ consists of CW–complexes, it follows that $X$ preserves weak equivalences, that is, $X$ is a homotopy functor.

We record here an important observation about objects of $\text{WTop}$. Since $\text{Id}_*$ is the basepoint of $W(*, *)$, the map $\text{Id}_{X(*)} = X(\text{Id}_*)$ is the base point of $\text{Top}(X(*), X(*))$. Hence $X(*) = *$ for any $X \in W$. We therefore say that every functor of $\text{WTop}$ is **reduced**.

**Remark 2.7** The category $W$ has a small skeleton, which fixes set-theoretic problems with the totality of natural transformations between functors from $\text{Top}$ to $\text{Top}$. In particular, it ensures that all small limits exist in $\text{WTop}$. The paper \cite{BR13} works with the simplicial analogue of $\text{WTop}$ and considers Goodwillie calculus in terms of simplicial functors from the category of finite simplicial sets $S^f$ to the category of all simplicial sets $S$.

A nice discussion of the set-theoretic problem can be found on Chorny-Dwyer \cite[Page 1]{CD09}, who circumvent this problem by placing a restriction on the type of functors used instead of on the source category. This idea is used in \cite{BCR07} to study Goodwillie calculus.

We now want to equip the category $\text{WTop}$ with a model structure, the following result is taken from \cite[Theorem 6.5]{MMSS01}.

**Lemma 2.8** The **projective model structure** on the category $\text{WTop}$ has fibrations and weak equivalences which are defined objectwise in the $q$-model structure of spaces. The cofibrations are determined by the left lifting property. This model structure is proper, cofibrantly generated and topological. The generating sets are given below, where $\text{sk}W$ denotes a skeleton of $W$.

$$I_{\text{WTop}} = \{W(X, -) \wedge i \mid i \in I_{\text{Top}}, X \in \text{sk}W\}$$

$$J_{\text{WTop}} = \{W(X, -) \wedge j \mid j \in J_{\text{Top}}, X \in \text{sk}W\}$$

One feature to note about this model structure is that the cofibrations are not in general objectwise $q$-cofibrations of spaces, since $W(A, B)$ is not usually $q$-cofibrant. However $W(A, B)$ is $m$-cofibrant (it is non-degenerately based and of the homotopy type of a CW-complex). Hence the projective cofibrations of $\text{WTop}$ are objectwise $m$-cofibrations and thus are objectwise $h$-cofibrations. The latter of these two facts will be very useful when making new model structures on $\text{WTop}$.
We claim that the model category $\mathcal{W} \text{Top}$ is a good model for homotopy functors from $\text{Top}$ to itself. To justify this, recall from [Goo03, Definition 5.10] that a homotopy functor from $\text{Top}$ to $\text{Top}$ is said to be finitary if it commutes with filtered homotopy colimits. This condition is clearly important for calculating the behaviour of a functor and [Goo03] often restricts attention to finitary homotopy functors.

In particular, if $F : \text{Top} \to \text{Top}$ is a finitary homotopy functor, then it is determined by its restriction to $\mathcal{W}$. Since any space $A$ is weakly equivalent to a homotopy colimit of finite CW-complexes $\text{hocolim}_n A_n$, we can extend a homotopy functor $X \in \mathcal{W} \text{Top}$ by the formula $X(A) = \text{hocolim}_n X(A_n)$ to obtain a finitary homotopy functor from $\text{Top}$ to itself.

To relate $\mathcal{W} \text{Top}$ to the work of Biedermann and Röndigs, consider the category of simplicial functors from the category of finite based simplicial sets to the category of based simplicial sets, $\text{Fun}(S^f, S)$. This category can be equipped the homotopy functor model structures of [BR13, Section 4]. It is then an exercise left to the enthusiast to show that $\mathcal{W} \text{Top}$ with its projective model structure is Quillen equivalent to $\text{Fun}(S^f, S)$ with the homotopy functor model structure. Hence the justifications of the reference for using $\text{Fun}(S^f, S)$ also apply to our setting.

3 Model structures for Goodwillie calculus

In this section, we first give background on excisive functors and the cross effect, then proceed to the related model structures.

3.1 Excisive functors and the cross effect

We now introduce some of the basic definitions of Goodwillie calculus, including excisive functors and the cross effect. We begin with necessary definitions to discuss excisive functors.

Definition 3.1 Let $\mathbb{n}$ denote the set $\{1, \ldots, n\}$ and let $\mathcal{P}(\mathbb{n})$ denote the powerset of $\mathbb{n}$. We define $\mathcal{P}_0(\mathbb{n})$ as the subset of $\mathcal{P}(\mathbb{n})$ consisting of non–empty subsets of $\mathbb{n}$.

We can give $\mathcal{P}(\mathbb{n})$ the structure of a poset by ordering the sets under inclusion. We may therefore think of this poset as the $2^n$ vertices of an $n$–dimensional cube. In turn $\mathcal{P}_0(\mathbb{n})$ is the cube with the initial point removed. As a poset, it naturally has the structure of a category, with objects the subsets and maps the inclusions.

Definition 3.2 An $n$–cube in $\mathcal{W}$ (or $\text{Top}$) is a functor $X$ from $\mathcal{P}(\mathbb{n})$ to $\mathcal{W}$ (resp. $\text{Top}$). An $n$–cube is said to be strongly cocartesian if all of its two-dimensional faces are homotopy pushout squares. An $n$–cube is said to be cartesian if the map

$$X(\emptyset) \to \text{holim}_{S \in \mathcal{P}_0(\mathbb{n})} X(S)$$

induced by the maps $X(\emptyset) \to X(S)$ is a weak equivalence.

Definition 3.3 An object $F \in \mathcal{W} \text{Top}$ is said to be $n$–excisive if it sends strongly cocartesian $(n + 1)$–cubes in $\mathcal{W}$ to cartesian $(n + 1)$–cubes in $\text{Top}$.

The first step in constructing the homotopy-universal approximation to $F$ by an $n$-excisive functor (denoted $P_n F$) is to define the functor $T_n f$. We note that we use $X * Y$ to denote the topological join of $X$ and $Y$.  

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Definition 3.4 We define a functor $T_n: W\text{Top} \to W\text{Top}$. For $F \in W\text{Top}$, $T_n F$ is given below.

$$(T_n F)(X) = \text{holim}_{S \in \mathcal{P}_0(n+1)} F(S \ast X)$$

The inclusion of the empty set as the initial set of the cube $\mathcal{P}_0(n + 1)$ and the fact that $\emptyset \ast X \cong X$ gives us a natural transformation $F(\emptyset \ast -) = F(-)$ to the homotopy limit $T_n F$, $t_{n,F}: F \to T_n F$.

For $F \in W\text{Top}$ we define

$P_n F = \text{hocolim}_k (F t_{n,F} \to T_n F t_{n,TnF} \to T_2^n F t_{n,T_2^n F} \to \ldots)$

The proof of [Goo03, Theorem 1.8] implies the following result.

Lemma 3.5 An object $F \in W\text{Top}$ is $n$-excisive if and only if the map $t_{n,F}: F \to T_n F$ is a weak equivalence for each $X \in W$.

We may now define the homotopy functors we are most interested in studying (and classifying): the $n$-homogeneous functors.

Definition 3.6 An object $F \in W\text{Top}$ is said to be $n$-homogeneous if it is $n$-excisive and $P_{n-1} F(X)$ is weakly equivalent to a point for each $X \in W$.

With the notion of $n$-excisive defined, we turn to the cross effect. The cross effect of a functor can be described as the fibre of a particular map, or as a space of natural transformations. We introduce the maps we need to give both descriptions.

For each integer $n \geq 1$ and each $n$-tuple $\underline{X} = (X_1, \ldots, X_n)$ of objects of $W$ we have a map

$$\phi_{\underline{X},n}: \text{colim}_{S \in \mathcal{P}_0(\underline{X})} W(\bigvee_{l \in \underline{n} - S} X_l, -) \to W(\bigvee_{l=1}^n X_l, -)$$

induced by the projections

$$\bigvee_{l=1}^n X_l \to \bigvee_{l \in \underline{n} - S} X_l$$

which acts as the identity on those factors with $l \notin S$ and sends the factors in $S$ to the basepoint. The map $\phi_{\underline{X},n}$ fits into the pushout square below, a proof is given in Section 3.2.

$$\text{colim}_{S \in \mathcal{P}_0(\underline{X})} W(\bigvee_{l \in \underline{n} - S} X_l, -) \xrightarrow{\phi_{\underline{X},n}} W(\bigvee_{l=1}^n X_l, -) \xrightarrow{\ast} \bigwedge_{l=1}^n W(X_l, -)$$

Now fix some $F \in W\text{Top}$ and apply the functor $\text{Nat}(-, F)$ (and the Yoneda lemma) to the above pushout square to obtain a pullback square as below.

$$\text{Nat}(\bigwedge_{l=1}^n W(X_l, -), F) \to F(\bigvee_{l=1}^n X) \xrightarrow{\phi_{\underline{X},n}} \lim_{S \in \mathcal{P}_0(\underline{X})} F(\bigvee_{l \in \underline{n} - S} X_l)$$
Definition 3.7 For $F \in W_{Top}$ and an $n$-tuple of spaces in $\mathcal{W}$, $(X_1, \ldots, X_n)$, the $n$th cross effect of $F$ at $(X_1, \ldots, X_n)$ is the space

$$\text{cr}_n(F)(X_1, \ldots, X_n) = \text{Nat}(\bigwedge_{l=1}^{n} \mathcal{W}(X_l, -), F)$$

Pre-composing $\text{cr}_n(F)$ with the diagonal map $\mathcal{W}(X, Y) \to \bigwedge_{l=1}^{n} \mathcal{W}(X, Y)$ yields an object of $W_{Top}$ which we call $\text{diff}_n(F)$, which in keeping with language of orthogonal calculus, is the $n$th derivative. That is

$$\text{diff}_n(F)(X) = \text{Nat}(\bigwedge_{l=1}^{n} \mathcal{W}(X, -), F)$$

Remark 3.8 We caution the reader that, as in [BR13], there is a difference between what Goodwillie [Goo03] calls the $n$th cross-effect and the above notation. [BR13] resolves this by calling Goodwillie’s version the homotopy cross-effect. If we need to refer to it, we will do the same.

3.2 The cross effect model structure

Unfortunately, the projective model structure on $W_{Top}$ (given in Lemma 2.8) is not sufficient for our purposes, since we will want the cross effect to be a right Quillen functor. That is, if $f : F \to G$ is a fibration of $W_{Top}$, we want $\text{cr}_n(f) : \text{cr}_n F \to \text{cr}_n G$ to be an objectwise fibration of $W_{Top}$. This does not hold for the projective model structure as explained in the introduction to [BR13] Section 3.3. However, if there is a topological model structure on $W_{Top}$ such that the following object is cofibrant

$$Z \mapsto \bigwedge_{l=1}^{n} \mathcal{W}(X, Z)$$

then the definition of a topological model category (Definition 2.1) would force $\text{cr}_n(f)$ to be an objectwise fibration whenever $f$ is.

We follow Biedermann and Röndigs [BR13] (albeit topologically rather than simplicially) and introduce another model structure on $W_{Top}$ that is Quillen equivalent to the projective model structure. This alternative model structure will be called the cross effect model structure.

The key idea is to take the maps $\phi_{X,n}$ and declare them to be cofibrations of our new model structure.

$$\phi_{X,n} : \text{colim}_{S \in \tau_{0}(\mathcal{W})} \mathcal{W}(\bigvee_{l \in \mathbb{R} - S} X_l, -) \to \mathcal{W}(\bigvee_{l=1}^{n} X_l, -)$$

Since cofibrations are preserved by pushouts, it will follow that the map below will be a cofibration in the new model structure.

$$* \to \bigwedge_{l=1}^{n} \mathcal{W}(X_l, -)$$

For practical reasons we will want to add a slightly larger set of maps to our cofibrations. This will ensure that the resulting model category is a topological model category and give us a little more control over the resulting fibrations. To define this set we need to introduce the notion of a pushout product.
Definition 3.9 Given $f : A \to B$ a map of based spaces and $g : X \to Y$ in $\mathcal{W}$, define $f \Box g$, the pushout product of $f$ and $g$ as the map below.

$$f \Box g : B \wedge X \bigvee_{A \wedge X} A \wedge Y \to B \wedge Y$$

Definition 3.10 Define the set $\Phi_\infty$ to be following set of maps.

$$\Phi_n = \{ \phi_{X, n} \mid X = (X_1, \ldots, X_n), X_i \in \text{sk} \mathcal{W} \}$$

$$\Phi_\infty = \bigcup_{n \geq 1} \Phi_n$$

We will declare the set of maps $\Phi_\infty \Box I_{\text{Top}}$ to be cofibrations in our new model structure. Since we are following [BR13], we shall give those bits of the argument which are new to our setting and leave those that follow from their arguments as references.

Theorem 3.11 There is a cofibrantly generated model structure on $\mathcal{W}_{\text{Top}}$ whose weak equivalences are the objectwise weak equivalences and whose generating sets are given by

$$I_{\mathcal{W}_{\text{cr}}} = \Phi_\infty \Box I_{\text{Top}}$$
$$J_{\mathcal{W}_{\text{cr}}} = \Phi_\infty \Box J_{\text{Top}}$$

We call the cofibrations of this model structure cross effect cofibrations and call the fibrations the cross effect fibrations. We write $\mathcal{W}_{\text{Top}}^{\text{cross}}$ for this model category and $\hat{f}_{\text{cross}}$ for its fibrant replacement functor.

Proof. We use the recognition theorem of Hovey [Hov99, Theorem 2.1.19]. It states that there is a cofibrantly generated model structure on a category $\mathcal{M}$ with $I$ as the set of generating cofibrations, $J$ as the set of generating trivial cofibrations, and $W$ as the subcategory of weak equivalences if and only if the following conditions are satisfied.

1. The subcategory $W$ has the two out of three property and is closed under retracts.
2. The domains of $I$ are small relative to $I$-cell.
3. The domains of $J$ are small relative to $J$-cell.
4. $J$-cell $\subseteq W \cap I$-cof.
5. $I$-inj $\subseteq W \cap J$-inj.
6. Either $W \cap I$-cof $\subseteq J$-cof or $W \cap J$-inj $\subseteq I$-inj.

In our case the objectwise weak equivalences clearly satisfy the two-out-of-three property and are closed under retracts.

The two smallness conditions ((ii) and (iii)) follow from three facts. Firstly, that the generating cofibrations and acyclic cofibrations are objectwise $h$-cofibrations. Secondly, that $\mathcal{W}(X, -) \wedge A$ is small with respect to the objectwise $h$-cofibrations whenever $A$ is a compact topological space, see [Hov99, Proposition 2.4.2]. Thirdly, that finite colimits (such as pushouts) of small objects are small, see [Hir03, Proposition 10.4.8].

The result then follows by Lemma 3.12, which proves that conditions (v) and the second part of (vi) hold, and Lemma 3.15, which proves that condition (iv) holds.

Lemma 3.12 A map in $\mathcal{W}_{\text{Top}}$ has the right lifting property with respect to $I_{\mathcal{W}_{\text{cr}}}$ if and only if it has the right lifting property with respect to $J_{\mathcal{W}_{\text{cr}}}$ and is an objectwise weak equivalence.


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In order to prove Lemma 3.15 we need to examine the maps $\phi_{X,n}$ a little more closely. We need to show that they are $h$-cofibrations of based spaces. We do so by building them using the monoidal structure of unbased spaces.

**Definition 3.13** Let $f: A \to B$ and $g: C \to D$ be maps of based spaces. Then $f$ and $G$ induce a map

$$f \boxtimes g: A \times D \coprod_{A \times C} B \times C \to B \times D$$

which we call the $\boxtimes$-**pushout-product** of $f$ with $g$. The basepoint of the domain is the point $(a_0, c_0)$, for $a_0$ the basepoint of $A$ and $c_0$ the basepoint of $C$.

An inductive cubical argument shows that

$$\phi_{X,n}: \text{colim}_{S \in \mathcal{P}_0} \mathcal{W}(\bigvee_{l \in \mathbb{N} - S} X_l, -) \to \mathcal{W}(\bigvee_{l=1}^{n} X_l, -)$$

is exactly the $n$-fold $\boxtimes$-pushout-product

$$(\ast \to \mathcal{W}(X_1, -)) \boxtimes \cdots \boxtimes (\ast \to \mathcal{W}(X_n, -))$$

Let $f: A \to B$ and $g: C \to D$ be maps of based spaces. Then there is a canonical homeomorphism of based spaces from the cofibre of $f \boxtimes g$ to the cofibre of $f$ smashed with the cofibre of $g$. Thus identifying $\phi_{X,n}$ in terms of the $\boxtimes$-pushout-product also gives a proof that the cofibre of $\phi_{X,n}$ is $\bigwedge_{l=1}^{n} \mathcal{W}(X_l, -)$.

**Lemma 3.14** For each $n \geq 1$ and each $n$-tuple $X = (X_1, \ldots, X_n)$ of objects in $\mathcal{W}$, the map $\phi_{X,n}$ is an objectwise $m$-cofibration. Hence, $\phi_{X,n}$ is an objectwise $h$-cofibration.

**Proof.** The based space $\mathcal{W}(X,Y)$ is $m$-cofibrant in the category of based topological spaces. This is precisely the statement that $\ast \to \mathcal{W}(X,Y)$ is an $m$-cofibration of unbased spaces. The model category of unbased topological spaces is a monoidal model category when equipped with the mixed model structure [Col06, Section 6], [MP12, Theorem 17.3.1]. Thus the map

$$\phi_{X,n} = (\ast \to \mathcal{W}(X_1,Y)) \boxtimes \cdots \boxtimes (\ast \to \mathcal{W}(X_n,Y))$$

is an $m$-cofibration of unbased spaces. Hence $\phi_{X,n}$ is an objectwise $m$-cofibration of based spaces. \[\square\]

**Lemma 3.15** Every map in $J_{\mathcal{W}_{cr}}$-cell is both an $I_{\mathcal{W}_{cr}}$-cofibration and an objectwise weak equivalence.

**Proof.** Every map in $\Phi_n$ is an objectwise $h$-cofibration by Lemma 3.14. Thus every map in $J_{\mathcal{W}_{cr}}$ is an objectwise acyclic $h$-cofibration of Top by [MP12, Corollary 17.1.2]. It follows by [Hov99, Lemma 2.4.8] that every map in $J_{\mathcal{W}_{cr}}$-cell is an objectwise acyclic $h$-cofibration of Top.

That the maps in $J_{\mathcal{W}_{cr}}$-cell are also $I_{\mathcal{W}_{cr}}$-cofibrations follows from the fact that every map of $J_{\mathcal{W}_{cr}}$ is a $I_{\mathcal{W}_{cr}}$-cofibration. \[\square\]

Theorem 3.11 also gives us the following useful result.

**Corollary 3.16** The cofibrant objects of the cross effect model structure are small with respect to the class of objectwise $h$-cofibrations.
Proof. The codomains of $I_{W_{cr}}$ are small with respect to the class of objectwise $h$–cofibrations. Hence every retract of an $I_{W_{cr}}$–cell complex is small with respect to this class by [Hir03, Propositions 10.4.7 and 10.4.8]. But the cofibrant objects are precisely the retracts of the $I_{W_{cr}}$–cell complexes so the result follows.

**Proposition 3.17** The cross effect model structure on $W_{Top}$ is proper.

**Proof.** Every cross effect cofibration is an objectwise $h$–cofibration. Similarly every cross effect fibration is an objectwise $q$–fibration. Since weak equivalences, limits and colimits are all defined objectwise, the result follows from the properness of $Top$ with respect to the $h$–model structure.

**Lemma 3.18** For $k: A \rightarrow B$ a cofibration of based spaces and $(X_1, \ldots, X_n)$ an $n$–tuple of objects of $W$, the map

$$\bigwedge_{l=1}^{n} W(X_l, -) \land k: \bigwedge_{l=1}^{n} W(X_l, -) \land A \rightarrow \bigwedge_{l=1}^{n} W(X_l, -) \land B$$

is a cross effect cofibration.

**Proof.** The map $i_0: * \rightarrow S^0$ is a cofibration of based spaces. Since $\phi_{X,n}^\ast = \phi_{X,n} \square i_0$, we see that $\phi_{X,n}^\ast$ is a cross effect cofibration. Since pushouts of cofibrations are cofibrations, the map $\alpha: * \rightarrow \bigwedge_{l=1}^{n} W(X_l, -)$ is also a cross effect cofibration. It follows that

$$\alpha \square i = \bigwedge_{l=1}^{n} W(X_l, -) \land i$$

is a cross effect cofibration.

**Lemma 3.19** If $F$ is a cross effect fibrant object of $W_{Top}$ then the $n$th homotopy cross effect of $F$ is given by the strict $n$th cross effect.

**Proof.** See [BR13, Lemma 3.24].

### 3.3 The $n$-excisive model structure

In this section we perform a left Bousfield localisation of the cross effect model structure on $W_{Top}$ to obtain the $n$-excisive model structure. The class of fibrant objects of this model structure will be the class of $n$-excisive objects of $W_{Top}$. The cofibrations will remain unchanged and the weak equivalences will be the $P_n$-equivalences. This section is related to [BR13, Section 5.1], and is analogous to [BO13, Section 6]. The primary difference to the work of Biedermann and Röndigs is that every functor in $W_{Top}$ is a homotopy functor. This simplifies the definition of $P_n$ substantially.

Recall from Definition 3.4 that we have a functor $T_n: W_{Top} \rightarrow W_{Top}$ defined as

$$(T_n F)(X) = \holim_{S \in \mathcal{P}(n+1)} F(S \ast X)$$

and a natural transformation $t_{n,F}: F \rightarrow T_n F$ induced by the maps $\emptyset \leftrightarrow S \in \mathcal{P}(n+1)$ (and, by extension, $- \ast X$ applied to them, which is $X \rightarrow S \ast X$), such that the $n$-excisive approximation to $F$ is given by

$$P_n F = \hocolim_k (F \xrightarrow{t_{n,F}} T_n F \xrightarrow{t_{n,T_n F}} T_n^2 F \xrightarrow{t_{n,T_n^2 F}} T_n^3 F \rightarrow \ldots)$$
We may then define a $P_n$–equivalence of $W\text{Top}$ to be a map $f: F \to G$ such that $P_n f$ is an objectwise weak equivalence.

To make a model structure where the $P_n$–equivalences are weak equivalences, it is necessary and sufficient to make the maps $t_{n,F}$ into weak equivalences. By the proof of [Goo03, Theorem 1.8], the map $t_{n,F}: F \to T_n F$ is a $P_n$-equivalence for any $F$. On the other hand, if there was a model structure on $W\text{Top}$ where the maps $t_{n,F}$ are weak equivalences for all $F$, then the map $F \to P_n F$ would be a weak equivalence. If the the objectwise weak equivalences were also contained in this new collection of weak equivalences, then the $P_n$-equivalences would also be weak equivalences.

We can replace the class of maps $t_{n,F}$ by a set of maps of spaces of natural transformations. The maps $X \to S^* X$ (used to give $t_{n,F}$) induce a morphism

$$\text{hocolim}_{S \in \mathcal{P}_0(n+1)} W(S^* X, -) \to W(X, -)$$

Which, after applying $\text{Nat}(-, F)$ and using the Yoneda lemma once more, gives us a map

$$F(X) \to \text{Nat}(\text{hocolim}_{S \in \mathcal{P}_0(n+1)} W(S^* X, -), F)$$

We therefore have the following

$$\text{Nat}(\text{hocolim}_{S \in \mathcal{P}_0(n+1)} W(S^* X, -), F) \simeq \text{holim}_{S \in \mathcal{P}_0(n+1)} \text{Nat}(W(S^* X, -), F) \simeq \text{holim}_{S \in \mathcal{P}_0(n+1)} F(S^* X) = (T_n F)(X)$$

Hence, to make a new model structure on $W\text{Top}$ where the weak equivalences are the $P_n$–equivalences we will make the set of maps below into weak equivalences.

$$S_n = \{s_{n,X}: \text{hocolim}_{S \in \mathcal{P}_0(n+1)} W(S^* X, -) \to W(X, -) \mid X \in W\}$$

There is a standard process for this, known as left Bousfield localisation, see Hirschhorn [Hir03] for a comprehensive account. Our model structures are not cellular, so we cannot apply those results directly. Instead we construct this localised model structure as a cofibrantly generated model structure via Hovey’s recognition theorem, [Hov99, Theorem 2.1.19].

In Section 3.4 we shall need to know that our model structure is right proper. Since we have a fibrant replacement functor already constructed for us—the functor $P_n$—we may use a theorem of Bousfield [Bou01] to conclude properness of our model structure; see Proposition 3.21. In this section we focus on the cofibrant generation and leave the use of [Bou01] till last, the opposite approach to [BR13] Section 5.

Our first step is to replace the set of maps $S_n$ by a set of objectwise $h$-cofibrations using the mapping cylinder construction. Let $f: A \to B$ be a map in Top. Define $Mf$, the mapping cylinder of $f$, to be the pushout of the following square where $i_0$ is the inclusion of $A$ into $A \times [0,1]_+$, $a \mapsto (a,0)$.

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{i_0} & & \downarrow{} \\
A \times [0,1]_+ & \xrightarrow{h} & Mf
\end{array}$$
We then have a map $k : A \to Mf$ the composite of $i_1 : A \to A \wedge [0,1]_+, a \mapsto (a,1)$ and $h$. We also have a retraction map $r : Mf \to B$ induced by the projection $A \wedge [0,1]_+ \to A$. The map $k$ is an $h$–cofibration and the map $r$ is a homotopy equivalence.

Applying this to our setting, for $s_{n,X} \in S_n$ let $k_{n,X}$ be the map from the domain of $s_{n,X}$ into $Ms_{n,X}$ induced $i_1$. Similarly let $r_{n,X} : Ms_{n,X} \to \mathcal{W}(X,-)$ be the retraction. Define

$$K_n = \{ k_{n,X} : hocolim_{S \in \mathcal{P}_0(n)}\mathcal{W}(S \ast X, -) \to Ms_{n,X} \mid X \in \mathcal{W} \}.$$

**Lemma 3.20** A map $f : F \to G$ has the right lifting property with respect to the set

$$J_{n-exs} = (\Phi_\infty \Box J_{Top}) \cup (K_n \Box I_{Top})$$

if and only if $f$ is a cross effect fibration and the square below is a homotopy pullback for all $X \in \mathcal{W}$.

\[
\begin{array}{ccc}
F(X) & \longrightarrow & (T_n F)(X) \\
\downarrow & & \downarrow \\
G(X) & \longrightarrow & (T_n G)(X)
\end{array}
\]

**Proof.** This kind of model category statement follows a standard pattern, see the proof of [BR13, Theorem 5.8] for details. 

We now give the equivalent of [BR13, Lemma 5.9]. We include the details as the analogous proof is not provided in [BR13].

**Corollary 3.21** A map $f : F \to G$ has the right lifting property with respect to the set $J_{n-exs}$ if and only if $f$ is a cross effect fibration and the square below is a homotopy pullback for all $X \in \mathcal{W}$.

\[
\begin{array}{ccc}
F(X) & \longrightarrow & (P_n F)(X) \\
\downarrow & & \downarrow \\
G(X) & \longrightarrow & (P_n G)(X)
\end{array}
\]

**Proof.** First assume that $f$ has the right lifting property with respect to $J_{n-exs}$. Then the statement follows from Lemma 3.20 and the fact that finite homotopy limits commute with filtered colimits up to weak equivalence. For the other direction, assume that

\[
\begin{array}{ccc}
X & \longrightarrow & P_n X \\
\downarrow f & & \downarrow P_n f \\
Y & \longrightarrow & P_n Y
\end{array}
\]

is a homotopy pullback. Applying $T_n$ to the square yields again a homotopy pullback that fits in the following commutative diagram:

\[
\begin{array}{ccc}
X & \longrightarrow & T_n X & \longrightarrow & P_n X \\
\downarrow (3) & & \downarrow (2) & & \downarrow (1) \\
Y & \longrightarrow & T_n Y & \longrightarrow & P_n Y \\
\end{array}
\]

Square (1) is trivially a homotopy pullback. Square (1) combined with Square (2) is a homotopy pullback, hence (2) is one. Square (2) combined with (3) is one by assumption, hence (3) is one.

\[\square\]
**Corollary 3.22** For $F \in \text{WTop}$, the map $F \to \ast$ has the right lifting property with respect to $J_n$–exs if and only if $F$ is cross-effect fibrant and $n$–excisive.

**Proof.** Apply Corollary 3.21 with $G = \ast$. 

Now we can show give our main result of this subsection, the existence of the $n$-excisive model structure. We have included the proof as our topological setting requires us to consider a few details which do not occur in [BR13, Theorem 5.8].

**Theorem 3.23** There is a cofibrantly generated model structure on $\text{WTop}$ whose weak equivalences are the $P_n$–equivalences and whose generating sets are given by

$$I_n\text{-exs} = \Phi_{\infty} \Box I_{\text{Top}} \quad J_n\text{-exs} = (\Phi_{\infty} \Box J_{\text{Top}}) \cup (K_n \Box I_{\text{Top}})$$

The cofibrations of this model structure are the cross effect cofibrations and the fibrations are called $n$–excisive fibrations. In particular, every $n$–excisive fibration is a cross effect fibration. The fibrant objects of this model structure are the cross effect fibrant $n$–excisive functors. We write $\text{WTop}_{n\text{-exs}}$ for this model category, which we call the $n$–excisive model structure.

**Proof.** We follow the recognition theorem of [Hov99, Theorem 2.1.19], see Theorem 3.11. To begin, we note that the weak equivalences have the two out of three property.

Next we must check that the generating sets admit the small object argument. Since the set of generating cofibrations are unchanged, we only need check that the domains of $J_n$–exs are small (in the sense of [Hov99, Definition 2.1.3]) with respect to $J_n$–exs–cell.

The maps in $J_n$–exs are objectwise $h$–cofibrations, so every map in $J_n$–exs–cell is an objectwise $h$–cofibration. Thus it will suffice to show that the domains of $J_n$–exs are small with respect to the class of objectwise $h$–cofibrations. Homotopy colimits of diagrams of cofibrant objects are cofibrant by [Hir03, Theorem 18.5.2]. Hence the domains of $J_n$–exs are cofibrant in the cross effect model structure. Corollary 3.16 implies that they are small with respect to the class of objectwise $h$–cofibrations and so we have the required smallness.

Now we turn to the lifting conditions. Let $f: F \to G$ be a map with the right lifting property with respect to $J_n$–exs that is also a $P_n$–equivalence. In particular, we have assumed that the map $(P_n F)(X) \to (P_n G)(X)$ is a weak equivalence of spaces for all $X \in W$. By our pullback square description of this lifting property, see Corollary 3.21 we see that $F(X) \to G(X)$ is a weak equivalence of spaces for all $X \in W$. Thus $f$ is an objectwise weak equivalence and a cross effect fibration. So it has the right lifting property with respect to $I_n$–exs.

For the converse, let $f$ have the right lifting property with respect to $I_n$–exs. Then $f$ is a cross effect fibration and an objectwise weak equivalence. It follows immediately that $f$ satisfies the pullback square condition and is a $P_n$–equivalence.

Finally we must show that transfinite compositions of pushouts of the maps in $J_n$–exs are cross effect cofibrations and $P_n$–equivalences. But this follows by [Hir03, Propositions 3.2.10 and 3.2.11].

**Proposition 3.24** The $n$–excisive model structure on $\text{WTop}$ is proper.

**Proof.** The functor $P_n$ satisfies the assumptions of [Bou01, Theorem 9.3] (as verified in [BR13, Theorem 5.8]). Hence there is a proper model structure on $\text{WTop}$ with weak equivalences the $P_n$–equivalences and cofibrations the cross effect cofibrations. This model structure is precisely our $n$-excisive model structure, so it is proper.
Note that every $n$–excisive functor in $W\text{Top}$ is objectwise weakly equivalent to a cross effect fibrant $n$–excisive functor.

**Lemma 3.25** Fibrant replacement in $W\text{Top}_{n\text{-exs}}$ is given by first applying the functor $P_n$ and then applying $\hat{f}_{cross}$, the fibrant replacement functor of $W\text{Top}_{\text{cross}}$.

**Proof.** For $F \in W\text{Top}$, $P_n F$ is $n$–excisive. Applying $\hat{f}_{cross}$ we obtain an objectwise weakly equivalent object $\hat{f}_{cross} P_n F$. This object is also $n$–excisive and is cross effect fibrant. Hence it is fibrant in $W\text{Top}_{n\text{-exs}}$. ■

### 3.4 The $n$–homogeneous model structure

Our next class of functors to study are those which are ‘purely’ $n$–excisive, that is, those $F$ such that $P_n F \simeq F$ but $P_{n-1} F \simeq \ast$. These are called $n$–homogeneous functors.

Similarly to [BO13, Section 6] and [BR13, Section 6] we perform a right Bousfield localisation of $W\text{Top}_{n\text{-exs}}$ in order to obtain a new model structure where the cofibrant-fibrant objects precisely the $n$–homogeneous objects which are fibrant and cofibrant in the cross effect model structure. Thus every $n$–homogeneous object of $W\text{Top}$ will be objectwise weakly equivalent to a cofibrant-fibrant object of this new model structure.

There are two cosmetic differences between this section and [BR13, Section 6]. Firstly, we use topological spaces instead of simplicial sets. Secondly, we want to describe the weak equivalences of this model structure as the $\text{diff}_n$–equivalences (rather than the $\text{cr}_n$–equivalences). That is, those maps $f$ such that $\text{diff}_n(P_n f)(X)$ is a weak equivalence of based spaces for all $X \in W$. Of course, these two classes of equivalences are the same by [Goo03, Proposition 5.8], hence using either one would give the same model structure.

**Definition 3.26** For $F \in W\text{Top}$, define $D_n F \in W\text{Top}$ as the homotopy fibre of $P_n F \to P_{n-1} F$.

Since $P_n$ and $P_{n-1}$ commute with finite homotopy limits, it follows that object $D_n F$ is $n$–homogeneous. The functor $D_n F$ is called the $n$–homogeneous approximation to $F$.

We must introduce some of the terminology of right Bousfield localisation. We begin by giving the set of objects that we will use to create our right localisation (compare this with the larger set of [BR13, Definition 6.2]).

$$M_n = \{ \bigwedge_{l=1}^{n} W(X,-) \mid X \in \text{sk } W \}$$

We define a map $f : F \to G$ to be an $M_n$–coequivalence (also known as an $M_n$–colocal equivalence or an $M_n$–cellular equivalence) if the following map induced by $f$ is a weak equivalence of based spaces for all $X \in W$:

$$\text{Nat}(\bigwedge_{l=1}^{n} W(X,-), \hat{f}_{n\text{-exs}} F) \longrightarrow \text{Nat}(\bigwedge_{l=1}^{n} W(X,-), \hat{f}_{n\text{-exs}} G)$$

Recall that $\text{Nat}$ defines our enrichment in based spaces:

$$\text{Nat}(-,-) : (W\text{Top})^{op} \times W\text{Top} \to \text{Top}$$

and that $\hat{f}_{n\text{-exs}}$ is the fibrant replacement functor in $W\text{Top}_{n\text{-exs}}$. 18
We note that usually the weak equivalences of a right Bousfield localisation are defined in terms of homotopy function complexes (which are simplicial sets), see Hirschhorn [Hir03, Chapter 17]. Since simplicial sets and topological spaces are Quillen equivalent, it suffices to just use the topological enrichment of \( \text{WTop}_{n-\text{exs}} \).

We define an \( M_n \)-\textit{colocal cofibration} to be a map in \( \text{WTop} \) with the left lifting property with respect to those maps which are both \( n \)-excisive fibrations and \( M_n \)-coequivalences. We say that \( A \in \text{WTop} \) is \( M_n \)-\textit{colocal} if every map \( f : F \to G \) which is an \( M_n \)-coequivalence induces a weak equivalence of spaces

\[
\text{Nat}(A, f_{n-\text{exs}} F) \to \text{Nat}(A, f_{n-\text{exs}} G)
\]

**Theorem 3.27** There is a right proper model structure on \( \text{WTop} \) whose fibrations are the \( n \)-excisive fibrations and whose weak equivalences are the \( M_n \)-coequivalences. We call this model structure the \( n \)-\textit{homogeneous model structure} and denote it by \( \text{WTop}_{n\text{-homog}} \). The cofibrant objects are the cross effect cofibrant \( M_n \)-colocal objects.

**Proof.** By Christensen and Isaksen [CI04, Theorem 2.6] this model structure exists and is right proper. We have used the fact that cofibrantly generated model categories (such as \( \text{WTop}_{n-\text{exs}} \)) always satisfy [CI04, Hypothesis 2.4].

We may now offer a slight improvement to the description of the \( M_n \)-coequivalences. By the end of this section we will have shown that this class of maps is equal to the class of \( D_n \)-equivalences – those maps \( f \) such that \( D_n f \) is an objectwise weak equivalence. In Section 6 we shall turn the construction \( \text{diff}_n \) into a Quillen functor. Taking our cue from that section we define \( \text{hodiff}_n \) as \( \text{diff}_n \circ \text{cross} \) (recall that \( f_{n-\text{exs}} = \text{cross} P_n f \)). For now, we may informally think of \( \text{hodiff}_n \) as the derived functor of \( \text{diff}_n \).

**Lemma 3.28** The class of \( M_n \)-coequivalences is equal to the class of \( \text{hodiff}_n \)-equivalences. Furthermore, in the model structure \( \text{WTop}_{n\text{-homog}} \), every object is weakly equivalent to an \( n \)-homogeneous functor.

**Proof.** By Lemma 3.29 we have a weak equivalence

\[
\text{Nat}(\bigwedge_{l=1}^n \text{W}(X, -), f_{n-\text{exs}} F) \simeq \text{Nat}(\bigwedge_{l=1}^n \text{W}(X, -), f_{\text{cross}} P_n F)
\]

But this is precisely \( \text{hodiff}_n F \). Hence the \( M_n \)-coequivalences are the \( \text{hodiff}_n \)-equivalences.

The functor \( \text{diff}_n \) is defined in terms of natural transformations out of the cofibrant object \( \bigwedge_{l=1}^n \text{W}(X, -) \), so it preserves homotopy fibre sequences. Furthermore \( \text{diff}_n P_n^{-1} F \simeq * \), since \( \text{diff}_n \) is given by the homotopy cross effect pre-composed with the diagonal, see [Goo03, Proposition 3.3]. Thus the maps \( D_n F \to P_n F \leftarrow F \) are weak equivalences in \( \text{WTop}_{n\text{-homog}} \).

Our theorem above shows that we have adjoint pairs as below. Furthermore, since we know that every object in \( \text{WTop}_{n\text{-homog}} \) is weakly equivalent to an \( n \)-homogeneous functor, we know that the composite derived functor \( \text{WTop}_{n\text{-homog}} \to \text{WTop}_{(n-1)\text{-exs}} \) sends every object to the trivial object.

\[
\text{WTop}_{n\text{-homog}} \underbrace{\to \text{WTop}_{n\text{-exs}} \underbrace{\to \text{WTop}_{(n-1)\text{-exs}}}}
\]

Our next task is to show that this behaves like an ‘exact sequence’ of model categories. That is, we want to show that an object \( F \) of the \( n \)-excisive model structure has \( P_{n-1} F \simeq * \) if and only if it is in the image of the derived functor from \( \text{WTop}_{n\text{-homog}} \) to \( \text{WTop}_{n\text{-exs}} \). We (roughly) follow the plan of [BR13, Section 6] by showing that the \( n \)-homogeneous model structure is stable. Whence we can work with spectrum-valued functors and so apply the results of [Goo03].
Lemma 3.29 The category of topological functors from \( W \) to (sequential) spectra \( \text{Sp} \) (denoted \( \text{Sp}[W\text{Top}] \)) admits a projective model structure, a cross effect model structure, an \( n \)-excisive model structure and an \( n \)-homogeneous model structure.

**Proof.** We may consider the category of sequential spectra in \( W\text{Top} \). The objects are collections of functors \( E_n \in W\text{Top} \), \( n \geq 1 \), with structure maps \( S^1 \wedge E_n \to E_{n+1} \). The morphisms are collections of maps of \( W\text{Top} \) that commute with the structure maps. The category of topological functors from \( W \) to \( \text{Sp} \) is equivalent to the category of sequential spectra in \( W\text{Top} \).

The above constructions of the various model structures apply to \( \text{Sp}[W\text{Top}] \). For example, the cross effect model structure is made by taking the generating cofibrations to be \( \Phi_\infty \Box I \text{Sp} \) and the generating acyclic cofibrations to be \( \Phi_\infty \Box J \text{Sp} \), where \( I \text{Sp} \) and \( J \text{Sp} \) are the generating sets for sequential spectra [MMSS01, Theorem 9.2].

There is an adjoint pair \( (\Sigma_\infty, \text{Ev}_0) \) between \( W\text{Top} \) and \( \text{Sp}[W\text{Top}] \). The left adjoint is \( \Sigma_\infty \), which is defined as usual: \( (\Sigma_\infty E)_n = S^n \wedge E \), for \( E \in W\text{Top} \). The right adjoint \( \text{Ev}_0 \) evaluates a spectrum at the zeroth level and its derived functor is \( \Omega_\infty \). The adjunction is a Quillen pair when both sides are given corresponding model structures, see [BR13, Lemma 6.12].

Proposition 3.30 The model category \( W\text{Top}_{n\text{-homog}} \) is stable.

**Proof.** We know that every object of \( W\text{Top}_{n\text{-homog}} \) is weakly equivalent to an \( n \)-homogeneous functor. Thus we may apply the results of [Goo03, Section 2] to obtain a delooping of an \( n \)-homogeneous functor. This delooping implies that \( (\Sigma_\infty, \text{Ev}_0) \) is a Quillen equivalence, as explained in [BR13, Theorem 6.11]. Since \( \text{Sp}[W\text{Top}] \) with the \( n \)-homogeneous model structure is stable, so is \( W\text{Top}_{n\text{-homog}} \).

Corollary 3.31 The \( n \)-homogeneous model structure \( W\text{Top}_{n\text{-homog}} \) is left proper and cofibrantly generated.

**Proof.** Left properness follows by looking at the cofibres of a pushout square and using stability, see [BR13, Proposition 5.8]. That this model category is cofibrantly generated is a small variation on [BR13, Theorem 5.9]. The generating cofibrations are given by the union of the generating acyclic cofibrations for \( W\text{Top}_{n\text{-exs}} \) along with the set of morphisms

\[
\{ S_k^+ \wedge W_n(X, -) \to D_k^+ \wedge W_n(X, -) \mid k \geq 0, \ X \in \text{sk} \ W \}. 
\]

Lemma 3.32 A map is a hodiff\( n \)-equivalence if and only if it is a \( D_n \)-equivalence.

**Proof.** Let \( f \) be a \( D_n \)-equivalence, so \( D_nf \) is an objectwise equivalence. Since hodiff\( n f = \text{diff}_n \hat{f}_{cross} P_n f \), it is weakly equivalent to \( \text{diff}_n \hat{f}_{cross} D_n f \), the first half of the result follows.

For the converse, we mostly follow [BR13, Lemma 6.19]. Take some hodiff\( n \)-equivalence \( f \). We can extend this to a map \( \Sigma_\infty f \) between functors to spectra. Since \( (\Sigma_\infty, \text{Ev}_0) \) is a Quillen equivalence, applying the homotopy cross effect pre-composed with the diagonal to \( \Sigma_\infty f \) gives an objectwise weak equivalence of spectra.

By [Goo03, Proposition 5.8] it follows that hocr\( n \Sigma_\infty f \) is an objectwise weak equivalence of spectra. The result [Goo03, Proposition 3.4] (see also [BR13, Corollary 6.9]) implies that \( D_n \Sigma_\infty f \) is also an objectwise weak equivalence. Hence so is \( \text{Ev}_0 D_n \Sigma_\infty f \). The functor \( \text{Ev}_0 \) commutes with \( D_n \) (up to objectwise weak equivalence) and \( \text{Ev}_0 \Sigma_\infty \simeq \text{Id} \). Thus \( D_n f \) is an objectwise weak equivalence.
Proposition 3.33 An object of $\mathcal{W}\text{Top}_{n\text{-homog}}$ is cofibrant and fibrant if and only if it is $n$-homogeneous and fibrant and cofibrant in the cross effect model structure. The cofibrations of $\mathcal{W}\text{Top}_{n\text{-homog}}$ are the cross effect cofibrations that are $P_{n-1}$- equivalences.

Proof. This follows from [BR13, Lemma 6.24].

Thus we now see that the cofibrant-fibrant objects of $\mathcal{W}\text{Top}_{n\text{-homog}}$ are exactly those functors of $\mathcal{W}\text{Top}_{n\text{-exs}}$ that are trivial in $\mathcal{W}\text{Top}_{(n-1)\text{-nexs}}$.

4 Capturing the derivative

We begin this section by giving a stable model structure for the category of spectra with a $\Sigma_n$-action (as these classify the $n$-homogeneous functors). This serves as a good warm up for defining the related category $\Sigma_n \ltimes (\mathcal{W}_n \text{Top})$ and giving it a stable model structure, Theorem 4.12. It plays the role analogous to the intermediate category $O(n)\mathcal{E}_n$ of Barnes-Oman, see [BO13, Section 7]. This category has been designed to receive Goodwillie’s derivative and we shall show in Section 6 that the derivative is part of a Quillen equivalence.

After defining the category $\Sigma_n \ltimes (\mathcal{W}_n \text{Top})$, we establish the projective (also called levelwise) model structure in Theorem 4.8, then left Bousfield localise to get the stable structure. This makes use of the definition of $n\pi_*$-isomorphisms (analogous to [BO13, Definition 7.7]).

We note again that the derivative appears only briefly in the work of [BR13] and is only developed in the case of functors from spaces to spectra. With our techniques, we will be able to study the derivative in the unstable setting.

4.1 A model category for spectra with a $\Sigma_n$-action

We need to introduce the model category of spectra with a $\Sigma_n$-action, as these classify the $n$-homogeneous functors. This section will also serve as a warm up for the next section, where we introduce the category $\Sigma_n \ltimes (\mathcal{W}_n \text{Top})$ that will receive the derivative.

The $n$-polynomial and $n$-homogeneous model structures have been defined on the category $\mathcal{W}\text{Top}$ (see Sections 3.3 and 3.4), so it makes sense to use a version of spectra that is defined in terms of $\mathcal{W}\text{Top}$ as well. For this, we follow [MMSS01], who place a stable model structure on $\mathcal{W}\text{Top}$, which makes it Quillen equivalent to the other models of the stable homotopy category. The change we make is to work $\Sigma_n$-equivariantly. We use $\Sigma_n \ltimes \mathcal{W}\text{Top}$ to denote the category of $\Sigma_n$-objects in $\mathcal{W}\text{Top}$ and $\Sigma_n$-equivariant morphisms.

We first note that there are several equivalent descriptions of the category we will be working with, of continuous functors from $\mathcal{W}$ to spaces with a $\Sigma_n$-action. These are:

$$\Sigma_n \ltimes \mathcal{W}\text{Top} = \Sigma_n \ltimes \text{Fun}(\mathcal{W}, \text{Top}) \cong \text{Fun}(\mathcal{W} \times \Sigma_n, \text{Top}) \cong \text{Fun}(\mathcal{W}, \Sigma_n \ltimes \text{Top}).$$

We have a free functor $(\Sigma_n)_+ \wedge - : \text{Top} \to \Sigma_n \ltimes \text{Top}$. We use this to construct a free model structure on $\Sigma_n \ltimes \mathcal{W}\text{Top}$ where the cofibrant objects are free.

Similar to Lemma 2.8, we have the following, which we state without proof:
Lemma 4.1 The projective model structure on the category $\Sigma_n \odot \mathcal{W} \mathcal{O} p$ has as generating sets (where $\text{sk}\mathcal{W}$ denotes a skeleton of $\mathcal{W}$):

\[
\begin{align*}
I_{\Sigma_n \odot \mathcal{W} \mathcal{O} p} &= \{ \mathcal{W}(X, -) \wedge (\Sigma_n)_+ \wedge i \mid i \in I_{\mathcal{W} \mathcal{O} p}, X \in \text{sk}\mathcal{W} \} \\
J_{\Sigma_n \odot \mathcal{W} \mathcal{O} p} &= \{ \mathcal{W}(X, -) \wedge (\Sigma_n)_+ \wedge j \mid j \in J_{\mathcal{W} \mathcal{O} p}, X \in \text{sk}\mathcal{W} \}.
\end{align*}
\]

A fibration (resp. weak equivalence) in this model structure is a $\Sigma_n$-equivariant map $f$ in $\mathcal{W} \mathcal{O} p$ such that each $f(X)$ is a $q$-fibration (resp. $q$-weak equivalence) of the underlying non-equivariant spaces. The cofibrations are determined by the left lifting property. We note that if $F \in \Sigma_n \odot \mathcal{W} \mathcal{O} p$ is cofibrant, then each $F(X)$ is a free $\Sigma_n$-space. This model structure is proper, cofibrantly generated and topological.

We now modify the projective model structure to obtain the stable model structure. We first show how to relate $\Sigma_n \odot \mathcal{W} \mathcal{O} p$ to sequential spectra. This relation allows us to define the stable equivalences.

Definition 4.2 Let $F \in \Sigma_n \odot \mathcal{W} \mathcal{O} p$ and $A \in \mathcal{W}$. Then we may define a spectrum $F[A]$ by the following construction

\[
F[A]_k := F(A \wedge S^k),
\]

where we have forgotten the $\Sigma_n$-action as well. The assembly maps provide the structure maps of $F[A]$ as well as maps $F[A] \wedge B \to F[A \wedge B]$. We call $F[S^0]$ the underlying spectrum of $F$.

Definition 4.3 A map $f : F \to G$ in $\Sigma_n \odot \mathcal{W} \mathcal{O} p$ is said to be a $\pi_\ast$-isomorphism if $f$ induces a $\pi_\ast$-isomorphism on the underlying spectra of $F$ and $G$.

We then have the following $\Sigma_n$-equivariant analogue of [MMSS01, Theorem 9.2], which we state without proof. Note that we are using the absolute stable model structure of [MMSS01, Section 17].

Lemma 4.4 There is a stable model structure on $\Sigma_n \odot \mathcal{W} \mathcal{O} p$. It is formed by left Bousfield localising the projective model structure at the set of maps below

\[
\{(\Sigma_n)_+ \wedge S^1 \wedge \mathcal{W}(A \wedge S^1, -) \to (\Sigma_n)_+ \wedge \mathcal{W}(A, -) \mid n \geq 0, A \in \text{sk}\mathcal{W}\}
\]

The cofibrations are the same as for the projective model structure and the weak equivalences are the $\pi_\ast$-isomorphisms. This model structure is cofibrantly generated, proper and topological. When we are considering $\Sigma_n \odot \mathcal{W} \mathcal{O} p$ with the stable model structure, we denote it by $\Sigma_n \odot \mathcal{W} \text{Sp}$.

4.2 Definition of $\Sigma_n \ltimes (\mathcal{W}_n \mathcal{O} p)$ and the projective model structure

Definition 4.5 Let $\mathcal{W}_n$ be the category enriched over topological spaces with objects those of $\mathcal{W}$ and spaces of morphisms given by

\[
\mathcal{W}_n(X, Y) := \bigwedge_{i=1}^{n} \mathcal{W}(X, Y)
\]

with the $\Sigma_n$-action which permutes the factors.
Definition 4.6 The category $\Sigma_n \ltimes (W_n \text{Top})$ is the category of $\Sigma_n \otimes \text{Top}$-enriched functors from $W_n$ to $\Sigma_n \otimes \text{Top}$. This category may also be written in the less compact form $\Sigma_n \text{Fun}(W_n, \Sigma_n \otimes \text{Top})$.

Thus a functor $X$ in the category $\Sigma_n \ltimes (W_n \text{Top})$ consists of the following information: a collection of based $\Sigma_n$-spaces $X(A)$ for each $A \in W_n$ and a collection of $\Sigma_n$-equivariant maps of based $\Sigma_n$-spaces 

$$X_{A,B} : W_n(A, B) \to \text{Top}(X(A), X(B))$$

for each pair $A$, $B$ in $W_n$. The $\Sigma_n$-structure on $\text{Top}(X(A), X(B))$ is given by conjugation: pre-composition with the inverse of an element and post-composition with that element. The maps $X_{A,B}$ must be compatible with composition and also associative and unital. The map $X_{A,B}$ induces a structure map, where $\Sigma_n$ acts diagonally on the smash product:

$$X(A) \wedge W_n(A, B) \to X(B).$$

Note that when $n = 1$, $\Sigma_n \ltimes (W_n \text{Top})$ is just $W\text{Top}$.

Remark 4.7 The idea for the notation $\Sigma_n \ltimes (W_n \text{Top})$ comes from the semi-direct product of groups. An object of the category $\Sigma_n \ltimes (W_n \text{Top})$ is in particular a topological functor from $W_n$ to $\text{Top}$. But we also have an action of $\Sigma_n$ on $W_n$ (which we elaborate on in Remark 5.2), so we note this by the symbol $\ltimes$.

Theorem 4.8 $\Sigma_n \ltimes (W_n \text{Top})$ has a projective (also called levelwise) model structure, starting with the free model structure on $\Sigma_n$-spaces. The generating cofibrations and trivial cofibrations are as follows

$$I_{W_n} = \{ W_n(A, -) \wedge (\Sigma_n)_+ \wedge i \mid i \in I_{\text{Top}}, A \in \text{sk } W \}$$

$$J_{W_n} = \{ W_n(A, -) \wedge (\Sigma_n)_+ \wedge j \mid j \in J_{\text{Top}}, A \in \text{sk } W \}$$

This defines a compactly generated topological proper model category denoted $\Sigma_n \ltimes (W_n \text{Top})_{\text{proj}}$.

Following [MM02, 2.4], we note that the proof is basically that of [MMSS01, 6.5].

4.3 The stable equivalences

We want to equip $\Sigma_n \ltimes (W_n \text{Top})$ with a stable model structure. To do this, we need to define the notion of $n\pi_*$-isomorphisms; note that $n = 1$ gives the usual definition of $\pi_*$-isomorphisms. Compare the following with [BO13, Definition 7.7] and Definitions 4.2 and 4.3.

Definition 4.9 The $n$-homotopy groups of an object $F$ of $\Sigma_n \ltimes (W_n \text{Top})$ at $A$ are denoted $n\pi^A_p(F)$ and defined as follows.

$$n\pi^A_p(F) := \colim_k \pi_p(\Omega^{nk} F(A \wedge S^k)) \cong \colim_k \pi_{p+nk}(F(A \wedge S^k))$$

A map is said to be an $n\pi^A_*$-isomorphism if it induces isomorphisms on $n\pi^A_k$ for all $k \in \mathbb{Z}$.

We establish independence of choice of space $A$ via the following result. Consequently, we may speak of $n$-homotopy groups and $n\pi_*$-isomorphisms without reference to a choice of space $A$. 
Proposition 4.10 A map \( f : F \rightarrow G \) in \( \Sigma_n \ltimes (W_n \text{Top}) \) is an \( n\pi^A_\ast \)-isomorphism for \( A = S^0 \) if and only if it is an \( n\pi^A_\ast \)-isomorphism for all \( A \in W \). We therefore call an \( n\pi^S_\ast \) isomorphism an \( n\pi_\ast \)-isomorphism.

Proof. This result follows by the same arguments as in [MMSS01, Proposition 17.6].

We now have the following analogue of the vitally important [MMSS01, Lemma 8.6]. This says that our \( n \)-stable equivalences are in particular \( n\pi_\ast \)-isomorphisms.

Corollary 4.11 The generalised evaluation maps

\[
\lambda_{A,n} : W_n(A \wedge S^1, -) \wedge S^n \rightarrow W_n(A, -)
\]

are \( n\pi_\ast \)-isomorphisms, as are the morphisms \((\Sigma_n)_+ \wedge \lambda_{A,n} \).

Proof. This follows from verifying that the following map is an isomorphism

\[
\colim_k \pi_{p+nk}(\Sigma^n \Omega^n W_n(A, S^k)) \rightarrow \colim_k \pi_{p+nk}(W_n(A, S^k)).
\]

This is simply an \( n \)-fold version of the \( \pi_\ast \)-isomorphism \( \Sigma \Omega X \rightarrow X \) for \( X \) a spectrum, so the result holds.

4.4 The stable model structure

The stable model structure on \( \Sigma_n \ltimes (W_n \text{Top}) \) is the left Bousfield localisation of the projective model structure at the set of maps

\[
(\Sigma_n)_+ \wedge \lambda_{A,n} : (\Sigma_n)_+ \wedge W_n(A \wedge S^1, -) \wedge S^n \rightarrow (\Sigma_n)_+ \wedge W_n(A, -)
\]

where \( S^n \), viewed as \( S^1 \wedge \cdots \wedge S^1 \), and \( W_n(A, -) \) have the \( \Sigma_n \)-action which permutes factors. Smash products are equipped with the diagonal action.

There are two problems: firstly that the general existence result on left Bousfield localisations requires our model structure to be cellular and secondly that it doesn’t give a set of generating (acyclic) cofibrations (which we will want later). But we can construct the stable model structure ourselves following [BO13, Section 7] and [MMSS01, Section 9] rather than using the machinery of [Hir03].

Proposition 4.12 The category \( \Sigma_n \ltimes (W_n \text{Top}) \) has a stable and proper model structure with cofibrations the projective cofibrations and whose weak equivalences are the \( n\pi_\ast \)-isomorphisms (of Definition 4.9). This model structure is denoted \( \Sigma_n \ltimes (W_n \text{Top})_{\text{stable}} \).

Analogous to [MMSS01, Prop 9.5], the fibrations are levelwise fibrations such that the square below is a homotopy pullback.

\[
\begin{array}{ccc}
F(A) & \rightarrow & \Omega^n F(A \wedge S^1) \\
\downarrow & & \downarrow \\
G(A) & \rightarrow & \Omega^n G(A \wedge S^1)
\end{array}
\]

The fibrant objects of this new model structure are the those \( F \) such that the maps \( F(A) \rightarrow \Omega^n F(A \wedge S^1) \) are weak equivalences for all \( A \in W \). An \( n\pi_\ast \)-isomorphism between fibrant objects is an objectwise weak equivalence.
The generating cofibrations are as in Theorem 4.8. The generating acyclic cofibrations are given by the generating acyclic cofibrations of that theorem in addition to the maps of equation 2

\[(\Sigma_n) + \land W_n(A \land S^1, -) \land S^n \rightarrow M((\Sigma_n) + \land \lambda_{A,n})\]

created as part of the mapping cylinder construction, for \(A \in sk W_n\).

**Proof.** This follows from [BO13] Section 7 and [MMSS01] Section 9. The most important step is Corollary 4.11 which implies that the weak equivalences are the \(n\pi_*\)-isomorphisms.

\[\square\]

5 Equivalence of the two versions of spectra

In this section we provide an adjunction between \(\Sigma_n \ltimes (W_n Top)\) and \(\Sigma_n \land WTop\). We then show that it is a Quillen equivalence. Thus we see that the stable model structure on \(\Sigma_n \ltimes (W_n Top)\) has the correct homotopy category to model \(n\)-homogeneous functors.

\[\Sigma_n \ltimes (W_n Top)_{stable} \xrightarrow{\phi_n^* \land \land} \Sigma_n \land WSp\]

We start by defining the right adjoint of this adjunction.

5.1 The adjunction between \(\Sigma_n \ltimes (W_n Top)\) and \(\Sigma_n \land Sp\)

**Definition 5.1** The functor \(\phi_n : W_n \rightarrow W\) is a Top-enriched functor. On objects it acts as \(X \mapsto X^{\land n}\) and on morphisms acts as the smash product

\[\begin{align*}
W_n(X,Y) & \rightarrow W(X^{\land n}, Y^{\land n}) \\
(f_1, \ldots, f_n) & \mapsto (f_1 \land \ldots \land f_n)
\end{align*}\]

This map of enriched categories \(\phi_n\) induces a functor \(\phi_n^*\), which is (almost) pre-composition with \(\phi_n\). Let \(F\) be an object of \(\Sigma_n \land WTop\). Then we define \((\phi_n^* F)(X) = F(X^{\land n})\), but with an altered action of \(\Sigma_n\). The space \(F(X^{\land n})\) has an action of \(\Sigma_n\) by virtue of \(F\) being a functor to \(\Sigma_n\)-spaces. We denote this action by \(\sigma \mapsto \sigma F(X^{\land n})\) and refer to it as the **external action**. The space \(X^{\land n}\) also has an action of \(\Sigma_n\), denoted \(\sigma_X\). We thus have a second action on \(F(X^{\land n})\), the **internal action**. We combine these and define the action on \((\phi_n^* F)(X)\) to be \(\sigma \in \Sigma_n \mapsto \sigma F(X) \cdot F(\sigma_X)\).

On mapping spaces, \((\phi_n^* F)_{X,Y}\) is given by the composite of the maps in the diagram below. We point out that an equivalent way of describing the space of \(\Sigma_n\)-equivariant maps from \(F(X^{\land n})\) to \(F(Y^{\land n})\) is \(Top(F(X^{\land n}), F(Y^{\land n}))^{\Sigma_n}\): fixed points with respect to the conjugation action.

\[\begin{array}{ccc}
W_n(X,Y) & \rightarrow & Top(F(X^{\land n}), F(Y^{\land n})) \\
\phi_n & \downarrow & \downarrow \\
W(X^{\land n}, Y^{\land n}) & \xrightarrow{F_{X^{\land n}, Y^{\land n}}} & Top(F(X^{\land n}), F(Y^{\land n}))^{\Sigma_n}
\end{array}\]

One needs to then verify that \(\phi_n^* F\) is \(\Sigma_n \land Top\)-enriched. This amounts to showing that \((\phi_n^* F)_{X,Y}\) is a \(\Sigma_n\)-equivariant map of spaces. It is clear that \((\phi_n^* F)_{X,Y}\) is continuous and straightforward to establish that it is \(\Sigma_n\)-equivariant. Hence we omit the proof.
Remark 5.2 We compare the different versions of equivariance for \( \Sigma_n \ltimes (W_n \text{Top}) \) and \( \Sigma_n \circ \text{Top} \).

Consider some \( F : W \rightarrow \Sigma_n \circ \text{Top} \). Then \( F(A) \in \Sigma_n \circ \text{Top} \) and for a map \( f \in W(A, B) \), the map \( F(f) : F(A) \rightarrow F(B) \) is \( \Sigma_n \)-equivariant. That is, \( F \) induces a map

\[
F_{A,B} : W(A, B) \rightarrow \text{Top}(F(A), F(B))^{\Sigma_n}.
\]

In contrast, for \( G \in \Sigma_n \ltimes (W_n \text{Top}) \), we have

\[
G_{A,B} : W(A, B)^{\wedge n} \rightarrow \text{Top}(G(A), G(B))
\]

is a \( \Sigma_n \)-equivariant map. Note that the altered action on \( (\phi_n^*F)(X) \) is required for the proof of Proposition 5.4.

The left adjoint \( W \wedge_{W_n} - \) takes an object \( F \) of \( \Sigma_n \ltimes (W_n \text{Top}) \) to the coend

\[
\int_{A \in W_n} F(A) \wedge W(A^{\wedge n}, -).
\]

The term \( W(A^{\wedge n}, -) \) has an action of \( \Sigma_n \) by permuting the factors of \( A^{\wedge n} \). Establishing the adjunction is a formal exercise in manipulating coends.

5.2 The Quillen equivalence

In this section we prove that the adjunction we have established is a Quillen equivalence.

\[
\Sigma_n \ltimes (W_n \text{Top})_{\text{stable}} \xrightarrow{W \wedge_{W_n} -} \Sigma_n \circ W \text{Sp}
\]

Lemma 5.3 The adjoint pair \( (W \wedge_{W_n} -, \phi_n^*) \) is a Quillen pair with respect to the stable model structures.

Proof. A generating cofibration of \( \Sigma_n \ltimes (W_n \text{Top}) \) is of the form \( W_n(A, -) \wedge (\Sigma_n)_+ \wedge i \), for \( i \) a cofibration of based spaces. The left adjoint sends this map to \( W(A^{\wedge n}, -) \wedge (\Sigma_n)_+ \wedge i \), which is a cofibration of \( \Sigma_n \circ W \text{Sp} \). Similarly, it sends the generating acyclic cofibrations of the projective model structure on \( \Sigma_n \ltimes (W_n \text{Top}) \) to acyclic cofibrations of \( \Sigma_n \circ W \text{Sp} \).

The stable model structure on \( \Sigma_n \ltimes (W_n \text{Top}) \) comes from taking the projective model structure and localising at the maps

\[
(\Sigma_n)_+ \wedge W_n(A \wedge S^1, -) \wedge S^n \rightarrow (\Sigma_n)_+ \wedge W_n(A, -)
\]

The left adjoint will take a map of the form above to the \( \pi_* \)-isomorphism

\[
(\Sigma_n)_+ \wedge W(A^{\wedge n} \wedge S^n, -) \wedge S^n \rightarrow (\Sigma_n)_+ \wedge W(A^{\wedge n}, -).
\]

It follows that the left adjoint is a left Quillen functor.

Proposition 5.4 The adjoint pair \( (W \wedge_{W_n} -, \phi_n^*) \) is a Quillen equivalence.
Proof. We claim that the right adjoint preserves all weak equivalences. A map \( f \) is a weak equivalence of \( \Sigma_n \circ W\text{Sp} \) if and only if \( f[S^0] \) is a \( \pi_* \)-isomorphism of spectra by [KMWSS01, Proposition 17.6]. Similarly a map \( g \) is a weak equivalence of \( \Sigma_n \times (W_n\text{Top}) \) if and only if it is an \( n\pi_* \)-isomorphism, by Proposition 4.12. By Proposition 4.10, \( g \) is an \( n\pi_* \)-iso if and only if \( n\pi^0_\ast \left( g \right) : n\pi^0_\ast (F) \to n\pi^0_\ast (G) \) is an isomorphism.

So consider \( \phi^*_n F \) for some object \( F \) in \( \Sigma_n \circ W\text{Sp} \). It is routine to check that

\[
n\pi^0_\ast (\phi^*_n F) = \text{colim}_k \pi_{p+nk} F(S^{nk}).
\]

By cofinality if follows that \( \phi^*_n f \) is an \( n\pi_* \)-isomorphism whenever \( f[S^0] \) is a \( \pi_* \)-isomorphism. Hence we have shown our claim that the right adjoint preserves all weak equivalences.

By [Hov99, Corollary 1.3.16], to complete the proof we must show that for cofibrant \( F \in \Sigma_n \times (W_n\text{Top}) \), the unit map of the adjunction

\[
F \longrightarrow \phi^*_n W \land_{W_n} F
\]
is a weak equivalence. Note that no fibrant replacements are needed in the above since the right adjoint preserves all weak equivalences. We can rewrite the unit map as

\[
F \cong \int_{A \in W_n} F(A) \land W(A, -)^{\land n} \longrightarrow \int_{A \in W_n} F(A) \land W(\phi_n(A), \phi_n(-)) \cong \phi^*_n W \land_{W_n} F.
\]

Note that the map above is only \( \Sigma_n \)-equivariant when \( \phi^*_n \) alters the action as described in its definition. Since the model category \( \Sigma_n \times (W_n\text{Top}) \) is stable, it suffices to check this in the case where \( F \) runs over a set of generators (in the sense of [SS03, Definition 2.1.2]).

It is routine to check that \( \Sigma_n \times (W_n\text{Top}) \) is generated by the object \( (\Sigma_n)_+ \land W_n(S^0, -) \). To see this, note that \([ (\Sigma_n)_+ \land W_n(S^0, -), F ]_+ \) is equal to \( n\pi_* (F) \). Replacing \( F \) by the generator and simplifying, we are left with the map below, which is induced by \( \phi_n \).

\[
(\Sigma_n)_+ \land W(S^0, -)^{\land n} \longrightarrow (\Sigma_n)_+ \land W(\phi_n(S^0), \phi_n(-)) = (\Sigma_n)_+ \land W(S^0, (-)^{\land n})
\]

This map is an isomorphism, since both domain and codomain are both given by the functor \( A \mapsto A^{\land n} \). Hence it is a weak equivalence as desired. \( \blacksquare \)

6 Differentiation is a Quillen equivalence

In this section we define differentiation as an adjunction between the homogeneous model structure on \( W\text{Top} \) and the stable model structure on \( \Sigma_n \times (W_n\text{Top}) \). We then show that it is a Quillen equivalence. Thus we will have a diagram of Quillen equivalences as below, showing that \( W\text{Top}_{n\text{-homog}} \) is Quillen equivalent to spectra with a \( \Sigma_n \)-action.

\[
\begin{array}{c}
W\text{Top}_{n\text{-homog}} \xrightarrow{(-)/\Sigma_n \circ \text{map-diag}^*} \Sigma_n \times (W_n\text{Top})_{\text{stable}} \\
\downarrow \text{diff}_n \quad \downarrow \phi^*_n \\
\Sigma_n \circ W\text{Sp}
\end{array}
\]

The first step is to show that differentiation is part of an adjunction as claimed. Then we show it is a Quillen pair between the cross effect model structure on \( W\text{Top} \) and the projective model structure on \( \Sigma_n \times (W_n\text{Top}) \). We next establish that this adjunction induces a Quillen adjunction between the \( n \)-excisive model structure and the stable model structure. Then we show that \(((-)/\Sigma_n \circ \text{map-diag}^*, \text{diff}_n) \) is a Quillen pair between the stable model structure and the \( n \)-homogeneous model structure. Finally we show that this last Quillen pair is indeed a Quillen equivalence.
6.1 The adjunction between $\Sigma_n \ltimes (\mathcal{W}_n\text{Top})$ and $\mathcal{W}\text{Top}^{n\text{-homog}}$

Recall Definition 3.7 where we define the $n^{th}$-cross effect. The $n^{th}$-derivative of $F$ is

$$\text{diff}_n(F)(X) = \text{Nat}(\bigwedge_{l=1}^{n} \mathcal{W}(X,-), F)$$

which is cross effect pre-composed with the diagonal. In that definition, we originally considered $\text{diff}_n(F)$ as an object of $\mathcal{W}\text{Top}$ by letting $\mathcal{W}(X,Y)$ act on $\text{diff}_n(X)$ by the diagonal map $\mathcal{W}(X,Y) \rightarrow \bigwedge_{l=1}^{n} \mathcal{W}(X,Y)$. We now consider $\text{diff}_n(F)$ as an object of $\Sigma_n \ltimes (\mathcal{W}_n\text{Top})$, by using composition of $\mathcal{W}_n$ to give a map

$$\bigwedge_{l=1}^{n} \mathcal{W}(X,Y) \wedge \text{diff}_n(F)(X) \rightarrow \text{diff}_n(F)(Y).$$

This map is $\Sigma_n$-equivariant, where the left hand side is given the diagonal action.

**Proposition 6.1** The functor $\text{diff}_n$ has a left adjoint:

$$(-)/\Sigma_n \circ \text{map-diag}^*: \Sigma_n \ltimes (\mathcal{W}_n\text{Top}) \rightarrow \mathcal{W}\text{Top}$$

which we define in the proof below.

**Proof.** We begin by defining the functor $\text{map-diag}: \mathcal{W} \rightarrow \mathcal{W}_n$. It is the identity on objects and the diagonal on morphisms:

$$f \in \mathcal{W}(A,B) \mapsto (f, \ldots, f) \in \mathcal{W}(A,B)^{\wedge n} = \mathcal{W}_n(A,B).$$

Let $E \in \Sigma_n \ltimes (\mathcal{W}_n\text{Top})$, then for $X \in \mathcal{W}$, $E(X)$ is a space with an action of $\Sigma_n$. Hence we define

$$((-)/\Sigma_n \circ \text{map-diag}^*(E))(X) = E(X)/\Sigma_n$$

We must also describe the structure maps of $((-)/\Sigma_n \circ \text{map-diag}^*(E)) \in \mathcal{W}\text{Top}$. It is the composite of the following three maps, the first is the diagonal map we have introduced above, the second takes $([x], (f, \ldots, f))$ to $[(x, f, \ldots, f)]$. The third is then the structure map of $E \in \Sigma_n \ltimes (\mathcal{W}_n\text{Top})$.

$$E(X)/\Sigma_n \wedge \mathcal{W}(X,Y) \rightarrow \frac{E(X)/\Sigma_n \wedge \bigwedge_{l=1}^{n} \mathcal{W}(X,Y)}{\Sigma_n} \rightarrow \frac{\left[E(X) \wedge \bigwedge_{l=1}^{n} \mathcal{W}(X,Y)\right]}{\Sigma_n} \rightarrow \frac{E(Y)/\Sigma_n}{\Sigma_n}$$
To see that we have an adjunction, we perform the following exercise in category theory.

\[
\begin{align*}
\mathcal{W}\text{Top}(E/\Sigma_n \circ \text{map–diag}, F) &= \int_{X \in \mathcal{W}} \text{Top}(E(X)/\Sigma_n, F(X)) \\
&= \int_{X \in \mathcal{W}} \text{Top}(E(X), \epsilon^* F(X))^{\Sigma_n} \\
&\approx \int_{X \in \mathcal{W}} \int_{Y \in \mathcal{W}_n} \text{Top}
\left(
E(Y) \wedge \bigwedge_{l=1}^n \mathcal{W}(Y, X), \epsilon^* F(X)
\right)^{\Sigma_n} \\
&\approx \int_{Y \in \mathcal{W}_n} \text{Top}
\left(
E(Y), \int_{X \in \mathcal{W}} \text{Top}
\left(
\bigwedge_{l=1}^n \mathcal{W}(Y, X), \epsilon^* F(X)
\right)
\right)^{\Sigma_n} \\
&\approx \int_{Y \in \mathcal{W}_n} \Sigma_n \text{Top}(E(Y), \text{diff}_n(F)(Y))^{\Sigma_n} \\
&= \Sigma_n \times (\mathcal{W}_n \text{Top})(E, \text{diff}_n(F)) \\
\end{align*}
\]

\[\blacksquare\]

**Lemma 6.2** The adjunction \((-)/\Sigma_n \circ \text{map–diag}^*, \text{diff}_n\) is enriched over based topological spaces.

**Proof.** There is an isomorphism, natural in \(E \in \Sigma_n \times (\mathcal{W}_n \text{Top})\) and \(K \in \text{Top}\)

\[(((E)/\Sigma_n \circ \text{map–diag}^*)(E)) \wedge K \rightarrow (((E)/\Sigma_n \circ \text{map–diag}^*)(E \wedge K))\]

induced by the isomorphism \(E(X)/\Sigma_n \wedge K \rightarrow (E(X) \wedge K)/\Sigma_n\). It follows that the right adjoint commutes with the cotensoring with \(\text{Top}\) and that the adjunction is enriched over topological spaces. \[\blacksquare\]

### 6.2 The Quillen equivalence

**Proposition 6.3** The adjunction \((-)/\Sigma_n \circ \text{map–diag}^*, \text{diff}_n\) is a Quillen pair with respect to the following pairs of model structures.

1. \(\Sigma_n \times (\mathcal{W}_n \text{Top})_{\text{proj}}\) and \(\mathcal{W}\text{Top}_{\text{cross}}\).
2. \(\Sigma_n \times (\mathcal{W}_n \text{Top})_{\text{proj}}\) and \(\mathcal{W}\text{Top}_{n-\text{exs}}\).
3. \(\Sigma_n \times (\mathcal{W}_n \text{Top})_{\text{stable}}\) and \(\mathcal{W}\text{Top}_{n-\text{exs}}\).
4. \(\Sigma_n \times (\mathcal{W}_n \text{Top})_{\text{stable}}\) and \(\mathcal{W}\text{Top}_{n-\text{homog}}\).

**Proof.** A projective cofibration of \(\Sigma_n \times (\mathcal{W}_n \text{Top})\) has the form below

\[\mathcal{W}_n(A, -) \wedge (\Sigma_n)_+ \wedge i\]

where \(i\) is a generating cofibration for based spaces. The functor \((-)/\Sigma_n \circ \text{map–diag}^*\) takes this to the map \(\mathcal{W}_n(A, -) \wedge i\) of \(\mathcal{W}\text{Top}\). This map is a cofibration of the cross effect model structure on \(\mathcal{W}\text{Top}\) by Lemma 3.18. Since the generating acyclic projective cofibrations have a similar form, it follows that \((-)/\Sigma_n \circ \text{map–diag}^*\) is a left Quillen functor as claimed in part (i). Part (ii) holds because the identity functor is a left Quillen functor from the cross effect model structure on \(\mathcal{W}\text{Top}\) to the \(n\)-excisive model structure on \(\mathcal{W}\text{Top}\).
For part (iii), by [Hir03, Theorem 3.1.6] we only need to show that \(\text{diff}_n\) takes fibrant objects of the \(n\)-excisive model structure to fibrant objects of the stable model structure. That is, if \(F\) is \(n\)-excisive and cross effect fibrant, then for any \(A \in W\)

\[
(\text{diff}_n F)(A) \to \Omega^n(\text{diff}_n F)(A \wedge S^1)
\]

is a weak equivalence. This is the content of [Goo03, Proposition 3.3] with the assumption that \(F(*)\) is equal to \(*\), rather than just weakly equivalent. This holds true for any object of \(\text{WTop}\) as is noted in Section 2.3. For part (iv), the cofibrations of the \(n\)-stable model structure have the form

\[
W_n(A, -) \wedge (\Sigma_n)_+ \wedge i
\]

Such a map is sent by \((-)/\Sigma_n \circ \text{map-diag}^*\) to the following map, which is a cofibration of the \(n\)-excisive model structure by Lemma 3.18.

\[
W_n(A, -) \wedge i
\]

This map is a cofibration of the \(n\)-homogeneous model structure by [Hir03, Proposition 3.3.16] and [Hir03, Lemma 5.5.2]. So the left adjoint preserves cofibrations.

We already know that the acyclic cofibrations of the stable model structure are sent to acyclic cofibrations of the \(n\)-excisive model structure. The acyclic cofibrations of the \(n\)-excisive model structure are exactly those of the \(n\)-homogeneous model structure. Hence the left adjoint preserves acyclic cofibrations.

**Theorem 6.4** The adjunction \((-)/\Sigma_n \circ \text{map-diag}^*, \text{diff}_n\) is a Quillen equivalence with respect to the \(n\)-stable model structure on \(\Sigma_n \ltimes (\text{W}_n \text{Top})\) and the \(n\)-homogeneous model structure on \(\text{W}_n \text{Top}\).

**Proof.** We show that the right adjoint reflects weak equivalences between fibrant objects. Let \(g: X \to Y\) be a map between cross effect fibrant \(n\)-excisive functors in \(\text{WTop}\) such that \(\text{diff}_n g\) is an \(n\pi_*\)-isomorphism in \(\Sigma_n \ltimes (\text{W}_n \text{Top})\). The domain and codomain of \(\text{diff}_n g\) are fibrant in the stable model structure by Proposition 6.3. Hence \(\text{diff}_n g\) is an objectwise weak equivalence by Proposition 4.12. The fibrancy assumption also tells us that \(g \simeq \hat{f}_{\text{exs}} g\), so that \(\text{diff}_n g\) is weakly equivalent to \(\text{hodiff}_n g\). Thus by Lemma 3.28 \(\text{diff}_n g\) is a weak equivalence of the \(n\)-homogeneous model structure.

We now show that for any cofibrant \(E \in \Sigma_n \ltimes (\text{W}_n \text{Top})\), the derived unit map

\[
E \to \text{hodiff}_n((-)/\Sigma_n \circ \text{map-diag}^*(E))
\]

is an \(n\pi_*\)-isomorphism. But this derived unit map is the map \(\theta\) of [Goo03, Theorem 3.5], which is an equivalence. Thus by [Hov99, Corollary 1.3.16] this adjunction is a Quillen equivalence.

**7 Quillen equivalence with symmetric multilinear functors**

We establish in Theorem 7.3 a Quillen equivalence between \(\Sigma_n \ltimes (\text{W}_n \text{Top})_{\text{stable}}\) and the category \(\text{Sym}–\text{Fun}(\text{W}_n, \text{Top})\) of symmetric functors with the symmetric-multilinear model structure. This result makes it clearer still that the category of symmetric multilinear functors can be omitted from the classification of \(n\)-homogeneous functors.

We begin by giving some definitions and recalling the statement of the symmetric-multilinear model structure of [BR13, Theorem 5.20]. Let \(\text{W}_n\) be the topological category with objects
\(n\)-tuples of spaces in \(W\) and morphisms spaces \(W(X_1, Y_1) \wedge \cdots \wedge W(X_n, Y_n)\) for \((X_1, \ldots, X_n)\) and \((Y_1, \ldots, Y_n)\) in \(W^n\). There are obvious functors between this category and \(W\) given by

\[
\begin{align*}
W & \longrightarrow W^n \\
X & \mapsto (X, \ldots, X) \\
f : X \rightarrow Y & \mapsto (f, \ldots, f)
\end{align*}
\]

\[
\begin{align*}
W^n & \longrightarrow W \\
(X_1, \ldots, X_n) & \mapsto X_1 \wedge X_2 \wedge \cdots \wedge X_n \\
(f_1, \ldots, f_n) & \mapsto f_1 \wedge f_2 \wedge \cdots \wedge f_n.
\end{align*}
\]

There is a less obvious functor from \(W_n\) to \(W^n\), which we call \(\text{ob–diag}\). It is the diagonal on objects and the identity on morphism spaces. That is, the space \(X\) is sent to \((X, \ldots, X)\) and the morphism space \(W_n(X, Y) = \bigwedge_{l=1}^n W(X, Y)\) is sent to itself by the identity. Recall the functor \(\text{map–diag}\), defined in the proof of Proposition 6.1. Note that \(\text{ob–diag} \circ \text{map–diag} = \Delta\), the normal diagonal functor for the smash product of spaces.

Let \(\text{Sym–Fun}(W^n, \text{Top})\) denote the category of symmetric functors from \(W^n\) to \(\text{Top}\). An \(n\)-variable functor \(F\) is symmetric precisely when, for each \(\sigma \in \Sigma_n\), there is a natural isomorphism \(F(X_1, \ldots, X_n) \cong F(X_{\sigma(1)}, \ldots, F(X_{\sigma(n)})\). When \(F\) is symmetric and \(X_l = X\) for all \(l\), \(F(X, \ldots, X)\) has an action of \(\Sigma_n\).

Using that action and pre-composition with \(\text{ob–diag}\) we obtain a functor from the category of symmetric functors on \(W^n\) to \(\Sigma_n \ltimes (W_n, \text{Top})\) which we call \(\text{ob–diag}^\ast\). We can also consider \(\text{cr}_n\) as a functor from \(\text{WTOP}\) to \(\text{Sym–Fun}(W^n, \text{Top})\). Since the cross effect precomposed with the diagonal is the functor \(\text{diff}_n\), we have the following commutative diagram of functors.

We use this diagram to relate our work and that of Biedermann and Röndigs [BR13]. In that paper they develop a symmetric multilinear model structure on the category of symmetric functors. This model structure is a modification of their hf ("homotopy functor")-model structure. Since we are working with functors in \(\text{WTOP}\), all of our functors are homotopy functors and the hf-model structure is then the projective model structure. We modify their statements (see [BR13, Definition 5.19]) accordingly:

**Theorem 7.1** There is a model category \(\text{Sym–Fun}(W^n, \text{Top})_{ml}\) whose underlying category is the category of symmetric functors from \(W^n\) to \(\text{Top}\).

- The weak equivalences are the maps \(f\) such that \(P_{1, \ldots, 1}(f)\) is an objectwise equivalence, called multilinear equivalences,
- the cofibrations are the projective cofibrations and
- the fibrations are the level wise fibrations \(f : F \rightarrow G\) such that the following square

\[
\begin{array}{ccc}
F & \longrightarrow & P_{1, \ldots, 1}F \\
\downarrow & & \downarrow P_{1, \ldots, 1}(f) \\
G & \longrightarrow & P_{1, \ldots, 1}G
\end{array}
\]

is a level wise homotopy pullback square.
Moreover, the fibrant objects are the symmetric multilinear functors.

In proving this result, it is helpful to have a different, but equivalent, description of the category. This alternate description (adjusted to our setting) is given below, see [BR13, Lemma 3.6].

**Definition 7.2** The **wreath product category** \((\Sigma_n \triangleright W^n)\) has objects the class of \(n\)-tuples \((X_1, \ldots, X_n)\) of objects of \(W\). The morphisms from \(X = (X_1, \ldots, X_n)\) to \(Y = (Y_1, \ldots, Y_n)\) are given by

\[
(\Sigma_n \triangleright W^n) (X, Y) = \bigvee_{\sigma \in \Sigma_n} \bigwedge_{l=1}^n W(X_l, Y_{\sigma^{-1}(l)})
\]

with composition is defined as it is in the wreath product of groups.

Given the model structure of Theorem 7.1, we may now establish the following formal comparison of our work with that of [BR13]. Note that the left adjoint to \(\text{ob–diag}^*\), denoted \(L_{\text{ob–diag}}\), sends \(F \in \Sigma_n \ltimes (W_n \text{Top})\) to the functor

\[
\int_{A \in W_n} F(A) \wedge \Sigma_n \triangleright W^n (\text{ob–diag}(A), -)
\]

in \(\text{Sym–Fun}(W^n, \text{Top})\).

**Theorem 7.3** The functor \(\text{ob–diag}^*\) is a right Quillen adjoint, and induces a Quillen equivalence between \(\text{Sym–Fun}(W^n, \text{Top})_{\text{ml}}\) and \(\Sigma_n \ltimes (W_n \text{Top})\) with the stable model structure.

**Proof.** Recall that in the stable model structure on \(\Sigma_n \ltimes (W_n \text{Top})\) (Proposition 4.12) the fibrations are those maps \(f : F \to G\) which are level wise fibrations, such that the following square is a homotopy pullback.

\[
\begin{array}{ccc}
F & \longrightarrow & \Omega^n F(- \wedge S^1) \\
\downarrow & & \downarrow \\
G & \longrightarrow & \Omega^n G(- \wedge S^1)
\end{array}
\]

The functor \(\text{ob–diag}^*\) sends level-wise (acyclic) fibrations to level-wise (acyclic) fibrations, and sends the square of Theorem 7.1 to the square above. Therefore it is a right Quillen functor. The functor \(\text{diff}_n\) is the right adjoint of a Quillen equivalence by Theorem 6.4 whereas \(\text{cr}_n\) is the right adjoint of a Quillen equivalence by [BR13, Corollary 6.17]. Since \(\text{ob–diag}^* \circ \text{cr}_n = \text{diff}_n\), it follows that \(\text{ob–diag}^*\) is also part of a Quillen equivalence.

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