THE LOCAL MULTIPLIER ALGEBRA: BLENDING NONCOMMUTATIVE RING THEORY AND FUNCTIONAL ANALYSIS

MARTIN MATHIEU

Dedicated to Robert Wisbauer on the occasion of his 65th birthday.

ABSTRACT. We discuss some basic features of the local multiplier algebra of a C^*-algebra, the analytic analogue of the well-known Kharchenko–Martindale symmetric ring of quotients, and also the more recent maximal C^*-algebra of quotients, which is the analytic companion to the Utumi–Lanning maximal symmetric ring of quotients, together with some of the applications to operator theory on C^*-algebras. The emphasis lies in illustrating the interrelations between noncommutative ring theory and functional analysis.

1. INTRODUCTION

Rings of quotients are a widely used concept in noncommutative ring theory and intimately connected with the important technique of localisation; see, e.g., [9]. Various 'breeds' have been developed depending on the kind of application one had in mind: for instance, the Kharchenko–Martindale symmetric ring of quotients of a semiprime ring serves particularly well in Galois theory [10] and in extending Herstein's programme on nonassociative derivations and isomorphisms from the simple case [6]. To a lesser extent analogous constructs have appeared in analysis, mainly due to the additional difficulties that arise from the need of complete spaces (*closed* ideals) and continuous mappings (*bounded* homomorphisms). In the early 1990's, together with Pere Ara, we started a systematic study of an analogue of the symmetric ring of quotients in the context of C^*-algebras, the local multiplier algebra, and more recently extended this to a more comprehensive treatment of C^*-algebras of quotients. Important precursors were papers by Elliott [7] and Pedersen [18] aiming at the structure of automorphisms and derivations of C^*-algebras; and, in fact, many of the uses of the local multiplier algebra are found in operator theory on C^*-algebras.

While this is documented in detail in our monograph [3], the purpose of the present article is to underline the similarities between the purely algebraic theory and its C^*-algebraic companion and to point out where modifications must be made. We shall also discuss briefly some of the applications of local multipliers to the structure theory of various classes of operators between C^*-algebras.

2000 *Mathematics Subject Classification.* 46L05; 16-06, 16A08, 16S90, 46-02, 46L07, 46L57.

Key words and phrases. Noncommutative rings of quotients, C^*-algebras, AW^*-algebras, local multiplier algebra, injective envelope, operator modules, derivations.

This paper is part of a research project supported by the Royal Society.

2. C^*-ALGEBRAS OF QUOTIENTS

For basic terminology and facts in C^*-algebra theory, we refer the reader to Section 1.2 of [3], where many further references can be found.

We begin by recalling the definition of a two-sided ring of quotients in a form which is especially well suited for our setting.

Definition 2.1. Let R be a semiprime ring with involution *. A unital ring S is called a *two-sided ring of quotients of* R if

(i) $R \subseteq S$;
(ii) $\forall\, b \in S$: $\; bJ + J^*b \subseteq R$ for some $J \in \mathfrak{I}$;
(iii) $\forall\, b \in S,\, J \in \mathfrak{I}$: $\; bJ = 0 \implies b = 0$;

where $\mathfrak{I}$ is a 'good' set of (right) ideals in R.

We shall not pause to spell out in detail the requirements on the 'good' set of ideals (since we will mainly be interested in two examples) nor discuss the properties the involution is supposed to have (such as positive-definiteness, e.g.), since in a moment all our rings will be C^*-algebras anyway.

Here are our main examples.

Examples 2.2. Let R be a semiprime ring with involution *.

1. If we choose $\mathfrak{I} = \{R\}$ then we obtain $S = M(R)$, the multiplier ring of R.
2. If we choose $\mathfrak{I} = \mathfrak{I}_e$, the set of all essential two-sided ideals of R, then we obtain $S = Q_s(R)$, the Kharchenko–Martindale symmetric ring of quotients of R.
3. If we choose $\mathfrak{I} = \mathfrak{I}_{er}$, the set of all essential right ideals of R, then we obtain $S = Q^s_{\max}(R)$, the Utumi–Lanning maximal symmetric ring of quotients of R.

Each of the above enjoys a universal property with respect to the prescribed set of ideals. Example 3 was already implicitly contained in Utumi's work in the 1950's but it was Lanning who started a systematic study [11]. Some explicit computations of $Q^s_{\max}(R)$ are carried out, e.g., in [16].

Following the above pattern we now introduce the concept of a C^*-algebra of quotients.

Definition 2.3. Let A be a C^*-algebra. A unital C^*-algebra B is called a *C^*-algebra of quotients of* A if

(i) $A \subseteq B$;
(ii) $\{b \in B \mid bJ + J^*b \subseteq A$ for some $J \in \mathfrak{I}\}$ is dense in B;
(iii) $\forall\, b \in B,\, J \in \mathfrak{I}$: $\; bJ = 0 \implies b = 0$;

where $\mathfrak{I}$ is a 'good' set of closed (right) ideals in A.

Evidently the main difference in the two concepts lies in condition (ii); the necessity of completeness forces us to make the adjustment in Definition 2.3.

As before we have three main examples in mind.

Examples 2.4. Let A be a C^*-algebra.

1. If we choose $\mathfrak{I} = \{A\}$ then we obtain $B = M(A)$, the multiplier algebra of A. This is a well studied C^*-algebra and the maximal *unitization* of A. It has found manifold applications in various areas of C^*-algebra theory.
2. If we choose $\mathfrak{I} = \mathfrak{I}_{ce}$, the set of all closed essential two-sided ideals of A, then we obtain $B = M_{\mathrm{loc}}(A)$, the local multiplier algebra of A. This C^*-algebra first came up in work by Elliott [7] and Pedersen [18] in the mid 1970's but seems to have lain dormant until the author and Pere Ara started a systematic investigation from 1990 onwards. Nowadays, its structure is fairly well understood and many uses have been found, see [3] and Section 4 below.
3. If we choose $\mathfrak{I} = \mathfrak{I}_{cer}$, the set of all closed essential right ideals of A, then we obtain $B = Q_{\max}(A)$, the maximal C^*-algebra of quotients of A. This algebra was first introduced in [1] and is now the topic of an ongoing research project by Ara and the author, see [5].

At this point we want to stress a subtle difference between Examples 2 and 3 in (2.4). For a two-sided ideal in a C^*-algebra, the concepts "closed essential" and "essential closed" coincide; that is to say if $I \subseteq A$ is a closed two-sided ideal which is essential as a closed ideal—$I \cap J \neq 0$ for every non-zero closed two-sided ideal $J \subseteq A$—then it is *algebraically* essential, that is, essential in A-$\mathcal{M}od$-A. The reason is that, in this case, "essential" can be expressed by an annihilator condition. This is not the case for one-sided, say right, ideals in general. So, a priori, a closed right ideal in a C^*-algebra which is essential in $\mathcal{B}an$-A, the category of Banach A-right modules, need not be essential in $\mathcal{M}od$-A. But, fortunately, it turns out that these concepts agree nevertheless. For this, and a comprehensive discussion of various notions of essentiality for one-sided ideals, see [5].

We use the chart below to discuss the interrelations between the purely algebraic and the analytic constructs.

$M_{\mathrm{loc}}(A)$	$Q_{\max}(A)$
$Q_s(A)$	$Q^s_{\max}(A)$
$Q_b(A)$	$Q^s_{\max}(A)_b$

The symmetric algebra of quotients $Q_s(A)$ of A of Definition 2.1 is endowed with a positive-definite involution, inherited from A, and a good order structure, thus allowing us to define *bounded elements* in the sense of Handelman–Vidav:

$$q \in Q_s(A) \ \text{ is bounded if } \ q^*q \leq \lambda 1 \ \text{ for some } \ \lambda \in \mathbb{R}_+; \tag{1}$$

in this case, the *norm* $\|q\|$ of q is defined to be $\sqrt{\lambda}$ for the least λ in (1) above. With this norm, the set of all bounded elements in $Q_s(A)$—the *bounded part* $Q_b(A)$ *of* $Q_s(A)$—becomes a pre-C^*-algebra, and its completion is $M_{\mathrm{loc}}(A)$. Consequently, $Q_s(A)$ and $M_{\mathrm{loc}}(A)$ contain a common $*$-subalgebra $Q_b(A)$; the latter is dense in

$M_{\mathrm{loc}}(A)$, and $Q_s(A)$ can be reconstructed from $Q_b(A)$ by central localisation. For more details on this, see [3, Section 2.2].

We call $Q_b(A)$ the *bounded symmetric algebra of quotients of* A.

A similar, if slightly more complicated mechanism creates a bounded part $Q^s_{\max}(A)_b$ in $Q^s_{\max}(A)$ and the completion of this pre-C^*-algebra is the *maximal C^*-algebra of quotients* $Q_{\max}(A)$ of A, see [1], [5]. Since we shall devote most of our attention to the local multiplier algebra, we will not discuss any further details at this point.

One way to construct the symmetric algebra of quotients $Q_s(A)$ is to employ (equivalence classes of) essentially defined double centralisers; that is, pairs of left and right module homomorphisms defined on essential ideals. These may not be continuous and hence may not be defined on closed ideals. The elements in $Q_b(A)$ correspond precisely to those which are defined via continuous (i.e., bounded) module homomorphisms, and, in general, $Q_b(A)$ is strictly smaller than $Q_s(A)$; see [3, Proposition 2.2.13].

The reader will also note that the two examples 2.2 (1) and 2.4 (1) agree with each other. This is a consequence of the fact that a right A-module homomorphism from A into A (or, more generally, from $I \in \mathfrak{I}_{ce}$ into A) is automatically continuous.

Since closed two-sided ideals in C^*-algebras are particularly well behaved, the above constructions can be performed in alternative ways; the one discussed in the next section is of fundamental importance for the applications.

3. The Local Multiplier Algebra

Suppose I and J are closed essential two-sided ideals in a C^*-algebra A. Then $I \cap J$ also belongs to $\mathfrak{I}_{ce}$ and, moreover, $I \cap J$ is an essential ideal in both $M(I)$ and $M(J)$. (This uses the property of closed ideals in a C^*-algebra to be idempotent.) By the universal property of the multiplier algebra, we obtain injective *-homomorphisms $M(I) \to M(I \cap J)$ and $M(J) \to M(I \cap J)$. These are given by "restricting the multipliers" to the smaller ideal. Since injective *-homomorphisms between C^*-algebras are isometries, we thus obtain a directed family of C^*-algebras $\{M(I) \mid I \in (\mathfrak{I}_{ce}, \supseteq)\}$ and *-monomorphisms $\{\rho_{JI} \mid I, J \in (\mathfrak{I}_{ce}, \supseteq)\}$, where $\rho_{JI}\colon M(I) \to M(J)$ is the above restriction homomorphism, if $J \subseteq I$. Taking the direct limit in the category of C^*-algebras yields the local multiplier algebra.

Definition 3.1. For a C^*-algebra A, $M_{\mathrm{loc}}(A) = \varinjlim_{\mathfrak{I}_{ce}} M(I)$ is the *local multiplier algebra* of A.

One of the basic achievements of our work with Ara was to show that this definition of the local multiplier algebra—which was first used in [7] and [18], but under a different name—agrees with the one presented in Section 2, thus providing the link between the algebraic and the analytic theory. See [3, Section 2.3].

In the case of C^*-algebras we can moreover describe the symmetric algebra of quotients in an alternative way.

Proposition 3.2. *For every C^*-algebra A, we have $Q_s(A) = \operatorname{alg}\varinjlim_{\mathfrak{I}_{ce}} M(K_I)$, where K_I denotes the Pedersen ideal of an ideal $I \in \mathfrak{I}_{ce}$.*

The *Pedersen ideal* of a closed two-sided ideal I in a C^*-algebra is the smallest two-sided ideal which is dense in I. If $I, J \in \mathfrak{I}_{ce}$ and $J \subseteq I$ then $K_J \subseteq K_I$ and $K_I = K_I^2$ is essential too. This enables us to prove the above result in [3, Proposition 2.2.4].

Examples 3.3. Let us look at some examples of local multiplier algebras.

(1) Let A be a commutative unital C^*-algebra; then $A = C(X)$, the complex-valued continuous functions on a compact Hausdorff space X. In this case, $M_{\mathrm{loc}}(A) = B(X)$, the algebra of all bounded Borel functions modulo the ideal of those functions that vanish off a rare subset of X. This algebra is sometimes called the *Dixmier algebra*, since in Dixmier's work it provided the first example of an AW^*-algebra which is not a von Neumann algebra (for $X = [0, 1]$). See [3, Proposition 3.4.5].
(2) A C^*-algebra A is called simple if it does not contain any closed two-sided ideal other than 0 and A. Evidently, for a simple C^*-algebra A, we have $M_{\mathrm{loc}}(A) = M(A)$. In [2] we gave examples of unital non-simple C^*-algebras A such that $M_{\mathrm{loc}}(A)$ is simple; hence $A \neq M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ in this case.
(3) A C^*-algebra A is said to be an *AW^*-algebra* if the left annihilator of every subset of A is principal, that is, of the form Ap for a projection $p \in A$. If A is an AW^*-algebra then $M_{\mathrm{loc}}(A) = A$; see, e.g., [3, Theorem 2.3.8]. Note that this in particular applies to every von Neumann algebra, that is, weakly closed unital C^*-subalgebra of the algebra $B(H)$ of all bounded linear operators on a Hilbert space H.
(4) Let $A = C(X) \otimes B(H)$ be the C^*-tensor product of $C(X)$, for a compact Hausdorff space X, and $B(H)$. (In this case, there is only one C^*-tensor norm on the algebraic tensor product.) Then

$$M_{\mathrm{loc}}(A) = \varinjlim_{U \in \mathfrak{D}} C_b(U, B(H)_s),$$

where $\mathfrak{D}$ is the filter of dense open subsets of X and $C_b(U, B(H)_s)$ denotes the C^*-algebra of all bounded continuous functions from U into $B(H)$ endowed with the strict topology. This result is proved in [5, Corollary 5.3].

The theory of the local multiplier algebra bears some similarity with the one of the symmetric ring of quotients but generally the additional analytic structure leads to complications. For instance, it is well known that Q_s is not a closure operation. For some time, it was an open problem, first raised in [18], whether M_{loc} is a closure operation or not, that is, whether there can be a C^*-algebra A such that $M_{\mathrm{loc}}(A)$ is different from $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$. This question was recently settled in [4].

Theorem 3.4. *There is a unital separable primitive approximately finite-dimensional C^*-algebra A such that $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \neq M_{\mathrm{loc}}(A)$.*

The proof of this result uses non-stable K-theory, Elliott's classification of AF-algebras and a detailed study of strict limits of sequences of projections in the local multiplier algebra, among others.

4. Applications of Local Multipliers

In this section we shall discuss some typical applications of local multipliers of C^*-algebras. The symmetric ring of quotients has been put to good use in the study of a number of classes of additive mappings on semiprime rings, notably automorphisms and derivations; see [6] and [10], for example. It is thus no surprise that the local multiplier algebra has too been exploited for this purpose.

One of Pedersen's original motivations to investigate multipliers of closed essential ideals of a C^*-algebra in [18] was to find a bigger C^*-algebra in which every derivation of the original C^*-algebra becomes inner, that is, is implemented as a commutator. A derivation d on a C^*-algebra A is a linear mapping $d\colon A \to A$ satisfying the usual Leibniz product rule; such a mapping is automatically bounded, as was first shown by Sakai. If there is an element a such that $dx = xa - ax$ for all $x \in A$, the derivation is called *inner*. In most cases such an element does not exist within A; therefore one tries to extend the derivation d to a bigger C^*-algebra which may contain an implementing element. Whether the local multiplier algebra has this property for every C^*-algebra is still unknown, though there have been some advances in this direction; see, e.g., [13] and [20]. Pedersen proved in [18] that the answer is positive if A is separable.

Once one knows that d is inner, one has a better chance to estimate its norm, which, of course, is important from the analytic point of view. It is easy to see that, if $dx = xa - ax$ for all $x \in A$, then $\|d\| \leq 2\,\mathrm{dist}(a, Z(A))$, where $Z(A)$ stands for the centre of A. In general, this estimate is strict and, in fact, a lower estimate is related to cohomological properties of A. Various kinds of C^*-algebras are known to have the property that the above inequality is indeed an equality for every inner derivation, such as von Neumann algebras, e.g. (a result by Zsido). We were able to show that this is true for local multiplier algebras and, more generally, for boundedly centrally closed C^*-algebras (see below for the definition). Moreover, Somerset proved that the distance $\mathrm{dist}(a, Z(A))$ from any element a to the centre is always attained, regardless of the nature of the C^*-algebra A [19].

Putting all this together one obtains full information on derivations on separable C^*-algebras with the aid of local multipliers; the details of the argument take up the major part of Sections 4.1 and 4.2 in [3].

Theorem 4.1 (Theorem 4.2.20 in [3])**.** *For every derivation d on a separable C^*-algebra A there exists $a \in M_{\mathrm{loc}}(A)$ such that $dx = xa - ax$ for all $x \in A$ and $\|d\| = 2\,\|a\|$.*

In the background of the above arguments to calculate the norm of an inner derivation works another category, the category of operator spaces together with completely bounded mappings (we shall briefly review this category in the next section), and the fact that the norm and the completely bounded norm of an inner derivation agree. The interplay with local multipliers becomes even more apparent when we now turn our attention to elementary operators.

Already in the mid 1950's Grothendieck proposed to use tensor products of Banach spaces to study operators defined between them. In the case of a C^*-algebra A, a very natural class arising in this way is the one consisting of elementary operators.

Define

$$\theta\colon M(A)\otimes M(A)\longrightarrow B(A),\quad a\otimes b\longmapsto M_{a,b},$$

where $M_{a,b}x = axb$ for $x \in A$, $a, b \in M(A)$ and $B(A)$ denotes the Banach algebra of all bounded linear operators on the C^*-algebra A with the operator norm. Elements in the image of θ are operators of the form $S\colon x \mapsto \sum_{j=1}^{n} a_j x b_j$, $a_j, b_j \in M(A)$ and are called *elementary operators* on A.

Once again it is easy to give upper estimates for the norm of an elementary operator S in terms of the norms of a_j and b_j but hard to give a precise description. To this end, Haagerup introduced a new norm on the tensor product of two C^*-algebras which is no longer a C^*-tensor norm but a good norm in the category of operator spaces. This so-called Haagerup norm is defined as follows. For $u \in M(A)\otimes M(A)$ put

$$\|u\|_h = \inf_{u=\sum_j a_j\otimes b_j}\Bigl\{\Bigl\|\sum_{j=1}^{n} a_j a_j^*\Bigr\|^{1/2}\Bigl\|\sum_{j=1}^{n} b_j^* b_j\Bigr\|^{1/2}\Bigr\},$$

the *Haagerup norm* of u. Then the following result holds.

Theorem 4.2. *For every infinite-dimensional simple unital C^*-algebra A, the mapping θ is an isometry on $A\otimes_h A$, the completion of the algebraic tensor product with respect to the Haagerup norm.*

This theorem is a special case of [3, Corollary 5.4.35] and rests heavily on results by Haagerup and Magajna. It is a prime example of a result in operator theory on C^*-algebras in the formulation of which local multipliers are absent—but in the proof of which they are essential. First of all we note that the assumption that A is unital is vital; otherwise the compact operators provide a counterexample. Under the hypothesis of Theorem 4.2, $M_{\mathrm{loc}}(A) = A$ and $Z(M_{\mathrm{loc}}(A)) = \mathbb{C}$; this is already important for the injectivity of θ! (As was first noticed in [12].) Furthermore the assumption entails that A is antiliminal and hence the operator norm of S and the completely bounded norm coincide ([3, Corollary 5.4.36]).

Once we move beyond a trivial ideal space, local multipliers become fully visible.

For a general C^*-algebra A with multiplier algebra $M(A)$ let ${}^{c}\!A = \overline{AZ}$ and ${}^{c}\!M(A) = \overline{M(A)Z}$ denote the *bounded central closure* of A and of $M(A)$, respectively, where $Z = Z(M_{\mathrm{loc}}(A))$ is the centre of the local multiplier algebra. (See also Section 5 below.) Both ${}^{c}\!A$ and ${}^{c}\!M(A)$ are modules over Z, and θ induces a mapping θ_Z on the module tensor product over Z. This module tensor product can be endowed with a central version of the Haagerup norm and will thus be denoted by ${}^{c}\!M(A)\otimes_{Z,h}{}^{c}\!M(A)$. Instead of $B(A)$ we now have to use $CB({}^{c}\!A)$ on the right hand side, the Banach algebra of all completely bounded operators on ${}^{c}\!A$ with the completely bounded norm (which, in general, is bigger than the operator norm). With this notation, the general result reads as follows.

Theorem 4.3 (Theorem 5.4.30 in [3])**.** *For every C^*-algebra A, the mapping θ_Z is an isometry from ${}^{c}\!M(A)\otimes_{Z,h}{}^{c}\!M(A)$ into $CB({}^{c}\!A)$.*

Other classes of operators that were studied in [3] with the help of local multipliers include generalised derivations, automorphisms, Jordan and Lie isomorphisms and centralising mappings, and results that use, but do not show, local multiplier theory (in the same vein as Theorem 4.2 above) are found, for instance, in the structure theory of Lie derivations on C^*-algebras; see [14].

5. Multipliers and the Injective Envelope

Lately, categories of operator spaces and completely bounded mappings have gained in importance in the study of C^*-algebras but also many other branches of infinite-dimensional analysis. One reason for this is the existence of injective hulls (which is not guaranteed in the category of C^*-algebras), which will lead us to a third picture of the local multiplier algebra in this section.

Just as the most general Banach space is $C(X)$, where X is a compact Hausdorff space,—since, by the Banach–Alaoglu theorem, every Banach space is isometrically isomorphic to a closed subspace of some space $C(X)$—the most general operator space is $B(H)$, the space of all bounded linear operators from H into itself, where H is a complex Hilbert space. Already in the early work by von Neumann in the 1920's it emerged that it is vital to consider matrices of operators as well; that is, matrices whose entries are from $B(H)$. Since such a matrix space $M_n(B(H))$ is isomorphic to $B(H^n)$, where H^n is the n-fold direct sum of H, $n \in \mathbb{N}$, there is a natural choice of a norm on $M_n(B(H))$; incidentally, the only one turning it into a C^*-algebra.

An abstract definition of operator spaces is provided by Ruan's axioms; see [17, Chapter 13]. However, as every C^*-algebra can be regarded as a concrete algebra of bounded linear operators on Hilbert space by the Gelfand–Naimark–Segal theorem, we can content ourselves with a concrete picture here.

Definition 5.1. An *operator space* E is a linear subspace of $B(H)$, for some Hilbert space H, such that, for each $n \in \mathbb{N}$, the space of matrices $M_n(E)$ is complete with respect to the canonical norm on $M_n(B(H)) = B(H^n)$.

The morphisms are given as follows.

Definition 5.2. For a linear mapping $T\colon E \to F$ between operator spaces E and F we denote by $T_n\colon M_n(E) \to M_n(F)$ the n-fold ampliation given by

$$T_n\big((x_{ij})_{1\le i,j\le n}\big) = \big(Tx_{ij}\big)_{1\le i,j\le n}.$$

The operator T is called *completely bounded* if $\|T\|_{cb} = \sup_{n\in\mathbb{N}} \|T_n\| < \infty$, where $\|T_n\|$ denotes the operator norm of each T_n. For a completely bounded operator the quantity $\|T\|_{cb}$ is called the *cb-norm* of T.

In the case where each T_n is an isometry, the operator T is called a *complete isometry*. If $\|T_n\| \le 1$ for all $n \in \mathbb{N}$ then T is said to be a *complete contraction*.

The two categories thus arising in a natural way are $\mathcal{O}_\infty$, consisting of operator spaces as the objects and completely bounded operators as the morphisms ("isomorphism" in this case means the existence of a bijective completely bounded linear mapping), and $\mathcal{O}_1$, consisting of operator spaces together with complete contractions; in this case, "isomorphism" is completely isometric linear isomorphism. We will work in the latter category, since two unital C^*-algebras are completely isometric if and only if they are isomorphic as C^*-algebras.

The notion of 'injective envelope' now is the usual one in a category.

Definition 5.3. An operator space I is called *injective* if, whenever $h\colon E \to F$ is a complete isometry between operator spaces E and F and $f_0\colon E \to I$ is a complete contraction, there exists $f\colon F \to I$, a complete contraction, such that $f \circ h = f_0$.

Definition 5.4. Let E be an operator space.

(i) A complete isometry $h\colon E \to F$ into another operator space F is said to be *essential* if, whenever $g\colon F \to G$ is a complete contraction into an operator space G such that $g \circ h$ is a complete isometry, then g is a complete isometry.

(ii) An injective operator space $I(E)$ together with an essential complete isometry $\iota\colon E \to I(E)$ is called an *injective envelope of* E.

Since, as usual, injective envelopes are unique up to isomorphism in $\mathcal{O}_1$, we will speak of *the* injective envelope of an operator space E and denote it by $I(E)$.

Wittstock was the first to show that $B(H)$ is injective in the above sense, and Hamana established the existence of the injective envelope and studied injective envelopes of C^*-algebras in great detail. For a nice exposition, see [17].

Theorem 5.5 (Hamana)**.** *For every C^*-algebra A, there is an injective envelope $I(A)$ which is a unital C^*-algebra containing A as a C^*-subalgebra. Moreover,*

(i) *$I(A)$ is monotone complete, hence an AW*-algebra;*

(ii) *if A is prime then $I(A)$ is prime, hence an AW*-factor;*

(iii) *if A is unital and simple then $I(A)$ is simple.*

The injective envelope of a C^*-algebra is, in general, a fairly big C^*-algebra, big enough to contain both the local multiplier algebra and the maximal C^*-algebra of quotients. For a C^*-algebra A, the *-subalgebra of $I(A)$

$$\{x \in I(A) \mid xI + Ix \subseteq A \ \text{ for some ideal } I \in \mathfrak{I}_{ce}\}$$

is isomorphic to $Q_b(A)$. This was first observed by Frank and Paulsen [8]; see also [5, Section 3]. Under this isomorphism we can, and will, identify $M_{\mathrm{loc}}(A)$ with a C^*-subalgebra of $I(A)$.

Additionally, the same kind of identification establishes an isomorphism between

$$\{y \in I(A) \mid yJ + J^*y \subseteq A \ \text{ for some right ideal } J \in \mathfrak{I}_{cer}\}$$

and $Q^s_{\max}(A)_b$. As $Q_{\max}(A) = \overline{Q^s_{\max}(A)_b}$, we can also consider $Q_{\max}(A)$ as a C^*-subalgebra of $I(A)$.

This leads to the following result; see Theorems 3.8 and 3.12 of [5].

Theorem 5.6. *For every C^*-algebra A, we have*

$$A \subseteq M(A) \subseteq M_{\mathrm{loc}}(A) \subseteq Q_{\max}(A) \subseteq \bar{A} \subseteq I(A)$$

and

$$Z(M_{\mathrm{loc}}(A)) = Z(Q_{\max}(A)) = Z(\bar{A}) = Z(I(A)).$$

The C^*-algebra $\overline{A}$ appearing in the above theorem is the *regular monotone completion* of A, defined as the smallest monotone complete C^*-subalgebra of $I(A)$ which contains A as an order-dense C^*-subalgebra. An immediate consequence of Theorem 5.6 is that $Q_{\max}(A) = A$ for every monotone complete C^*-algebra A, in particular, every von Neumann algebra.

The equality of all the centres of the C^*-algebras from $M_{\text{loc}}(A)$ upward in Theorem 5.6 is very interesting in itself. The local Dauns–Hofmann theorem [3, Theorem 3.1.1] describes the centre $Z = Z(M_{\text{loc}}(A))$ as a direct limit:

$$Z(M_{\text{loc}}(A)) = \varinjlim_{\mathfrak{I}_{ce}} Z(M(I))$$

and combining this with Theorem 5.6, we obtain a new concrete description of the centre of the injective envelope.

A C^*-algebra A is called *boundedly centrally closed* if $Z(M(A)) = Z$; equivalently, ${}^{c}A = A$ and ${}^{c}M(A) = M(A)$ [3, Proposition 3.2.3]. As a consequence of the local Dauns–Hofmann theorem, $M_{\text{loc}}(A)$ and, more generally, every C^*-subalgebra of $M_{\text{loc}}(A)$ containing both A and Z is boundedly centrally closed [3, Theorem 3.2.8]. Boundedly centrally closed C^*-algebras behave analogously to centrally closed semiprime rings (where the extended centroid agrees with the centroid of the ring R, that is, $Z(Q_s(R)) = Z(M(R))$. As a result many statements in operator theory on C^*-algebras become much simpler for this class of C^*-algebras.

It follows from the above that, for a unital boundedly centrally closed C^*-algebra A, $Z(A) = Z(I(A))$ although the injective envelope itself might be much bigger than A.

6. The Maximal C^*-Algebra of Quotients

Despite the fact that the maximal C^*-algebra of quotients $Q_{\max}(A)$ of a C^*-algebra A contains all the multiplier algebras of essential hereditary C^*-subalgebras of A, there does not seem to be a way to describe $Q_{\max}(A)$ as a direct limit of C^*-algebras, in contrast to the direct limit description of the local multiplier algebra; see Section 3. This makes the study of its properties much harder.

Fortunately there is at least a direct limit construction in the category of operator modules, which we will briefly indicate; for more details, see [5].

Given a unital C^*-algebra A we denote by $CB_A(E, F)$ the space of all completely bounded right A-module maps between the operator right A-modules E and F. If F is even an operator A-bimodule, this is an operator left A-module, where the operator space structure is given by $M_n(CB_A(E, F)) = CB_A(E, M_n(F))$, $n \in \mathbb{N}$ and the A-module structure on $CB_A(E, F)$ is defined by $(ag)(x) = ag(x)$, $a \in A$, $x \in E$ and $g \in CB_A(E, F)$.

For $I, J \in \mathfrak{I}_{cer}$, $J \subseteq I$, denote by $\rho_{JI} \colon CB_A(I, A) \to CB_A(J, A)$ the restriction map, which turns out to be a complete isometry. Letting

$$E_b(A) = \operatorname{alg}\varinjlim_{\mathfrak{I}_{cer}} CB_A(I, A)$$

we obtain an (uncompleted) direct limit in the category of operator left A-modules, which in fact is an (incomplete) unital operator algebra. The relation with the maximal C^*-algebra of quotients is given by the following result.

Theorem 6.1 ([5])**.** *For every C^*-algebra A, we can consider $E_b(A)$ as an operator subalgebra of $I(A)$. Under this identification, $E_b(A) \cap E_b(A)^* = Q^s_{\max}(A)_b$.*

At this stage, there are several basic questions still open for the maximal C^*-algebra of quotients.

Open Questions. 1. Is it true that $Q_{\max}(A) = A$ whenever A is an AW^*-algebra? We know that the answer is positive if A is finite; if A is σ-finite (countably decomposable); or if A is monotone complete [5], but not yet in general.

2. Is $Q_{\max}(A)$ always an AW^*-algebra? We expect the answer to be "no" but do not have a counterexample yet.

3. We know of examples such that $M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A)$, but is it possible that $Q_{\max}(Q_{\max}(A)) = Q_{\max}(A)$ for every C^*-algebra A?

New ways to obtain the maximal symmetric ring of quotients, as outlined in [15], e.g., may be useful to tackle these problems.

Acknowledgement. This paper is a slightly expanded version of a talk entitled *What can we say about local multipliers?* delivered on 8 September 2006 as part of the International Conference on Modules and Comodules dedicated to Robert Wisbauer and held at the Universidade do Porto. The support received from the *Deutsche Akademische Austauschdienst (DAAD)* to attend this meeting is gratefully acknowledged.

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Department of Pure Mathematics, Queen's University Belfast, Belfast BT7 1NN, Northern Ireland
E-mail address: m.m@qub.ac.uk