Normalisers, nest algebras and tensor products

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Abstract

We show that if $A$ is the tensor product of finitely many continuous nest algebras, $B$ is a CDCSL algebra and $A$ and $B$ have the same normaliser semi-group then either $A = B$ or $A^* = B$.

1 Introduction

Normalisers of selfadjoint operator algebras were introduced by Murray and von Neumann in the 1930’s and have played an important role in Operator Algebra Theory thereafter. They are used in a fundamental way in the theory of crossed products, a notion which provides a setting for Non-commutative Dynamics (see [13] and [20]). Normalisers constitute a basic object in the theory of limit algebras as well [16]. Normalisers of tensor products of von Neumann algebras were recently considered in [4], [11], [17] and [18]. The study of the normalisers of non-selfadjoint operator algebras, namely of nest algebras, was initiated in the 1990’s [1], [7], [5]. In [14] the notion of a normaliser was generalised and studied in the context of reflexive algebras, a non-selfadjoint generalisation of von Neumann algebras. It was shown that normalisers are closely related to ternary rings of operators, a class of spaces studied independently in Operator Space Theory (see [3]). This connection provided the base in [8] for the introduction of an equivalence relation for non-selfadjoint operator algebras which later lead to the study of a more general equivalence relation for abstract dual operator algebras linked to Morita equivalence [9], [10].

2000 Mathematics Subject Classification
Primary 47L35; Secondary 47D03

The first named author was supported by a grant from the Department for Employment and Learning of Northern Ireland.
If $\mathcal{A}$ is an operator algebra acting on a Hilbert space $H$, a **normaliser** of $\mathcal{A}$ is a bounded linear operator on $H$ such that

$$T^*\mathcal{A}T \subseteq \mathcal{A} \quad \text{and} \quad T\mathcal{A}T^* \subseteq \mathcal{A}.$$ 

Let $N(\mathcal{A})$ be the set of all normalisers of an operator algebra $\mathcal{A}$. It is obvious that $N(\mathcal{A})$ is a selfadjoint semi-group of operators containing the diagonal $\mathcal{A} \cap \mathcal{A}^*$ of $\mathcal{A}$ (here and in the sequel we let $\mathcal{A}^* = \{ A^* : A \in \mathcal{A} \}$). The question to what extent $N(\mathcal{A})$ determines $\mathcal{A}$ was considered in [19], where $\mathcal{A}$ was taken from the class of CSL algebras introduced by Arveson in his seminal work [2]. It is obvious that for any operator algebra $\mathcal{A}$ we have that $N(\mathcal{A}) = N(\mathcal{A}^*)$. It is thus natural to ask to what extent the converse is true; in particular, whether for two CSL algebras $\mathcal{A}$ and $\mathcal{B}$ the equality $N(\mathcal{A}) = N(\mathcal{B})$ implies that either $\mathcal{A} = \mathcal{B}$ or $\mathcal{A}^* = \mathcal{B}$. Within the classes where this holds one is able to determine (up to adjoint) the non-selfadjoint algebras belonging to the class by using selfadjoint objects, namely their normaliser semi-groups.

Easy examples, in which the atomic and the continuous parts of the invariant subspace lattices of the algebras are both non-trivial, show that this converse statement fails. It is however true if $\mathcal{A}$ and $\mathcal{B}$ are totally atomic CSL algebras, as well as when they are continuous nest algebras. Namely, the following result was established in [19]:

**Theorem 1.1** Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{A}$ and $\mathcal{B}$ be continuous nest algebras acting on $\mathcal{H}$. Suppose that $N(\mathcal{A}) = N(\mathcal{B})$. Then either $\mathcal{A} = \mathcal{B}$ or $\mathcal{A}^* = \mathcal{B}$.

A class of operator algebras larger than the class of nest algebras is that of CDCSL algebras. It has played an important role in non-selfadjoint operator algebra theory (see [6, Chapter 23]). CDCSL algebras are characterised among CSL algebras by the fact that the Hilbert-Schmidt operators contained in the algebra are weakly dense in it. We note that finite tensor products of nest algebras possess this property.

In this note we prove the following generalisation of Theorem 1.1:

**Theorem 1.2** Let $\mathcal{H}_i$ be a separable Hilbert space, $\mathcal{A}_i$ be a continuous nest algebra acting on $\mathcal{H}_i$, $i = 1, \ldots, n$, $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ and $\mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \subseteq \mathcal{B}(\mathcal{H})$. Suppose that $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ is a CDCSL algebra. If $N(\mathcal{A}) = N(\mathcal{B})$ then either $\mathcal{A} = \mathcal{B}$ or $\mathcal{A}^* = \mathcal{B}$.

The proof of Theorem 1.2 and some of its corollaries are given in Section 3. In the next section we collect preliminary notions and results.
2 Preliminaries

All Hilbert spaces in this note will be assumed to be separable. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the space of all bounded linear operators on $\mathcal{H}$. The set $\mathcal{S}(\mathcal{H})$ of all closed subspaces of $\mathcal{H}$ is a complete lattice with respect to intersection and closed linear span. Using the bijective correspondence between $\mathcal{S}(\mathcal{H})$ and the set $\mathcal{P}(\mathcal{H})$ of all orthogonal projections on $\mathcal{H}$, we can equip $\mathcal{P}(\mathcal{H})$ with a natural lattice structure. A **subspace lattice** on $\mathcal{H}$ is a sublattice $\mathcal{L} \subseteq \mathcal{P}(\mathcal{H})$ closed in the strong operator topology. Given a subspace lattice $\mathcal{L}$, we let

$$\text{Alg} \mathcal{L} = \{ A \in \mathcal{B}(\mathcal{H}) : (I - L)AL = 0, \text{ for each } L \in \mathcal{L} \}$$

be the algebra of all operators on $\mathcal{H}$ leaving every projection of $\mathcal{L}$ invariant. Obviously, $\text{Alg} \mathcal{L}$ contains the identity operator, and it is trivial to check that it is closed in the weak operator topology. Conversely, given a weakly closed unital subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$, we let

$$\text{Lat} A = \{ L \in \mathcal{P}(\mathcal{H}) : (I - L)AL = 0, \text{ for each } A \in A \}$$

be the lattice of all projections on $\mathcal{H}$ invariant under every operator in $A$. The set $\mathcal{L}$ is easily seen to be a subspace lattice. A weakly closed unital subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$ is called **reflexive** if $A = \text{Alg} \text{Lat} A$. By virtue of von Neumann’s Bicommutant Theorem, the class of reflexive algebras contains all von Neumann algebras.

A **commutative subspace lattice (CSL)** on $\mathcal{H}$ is a subspace lattice $\mathcal{L} \subseteq \mathcal{P}(\mathcal{H})$ with the property that $PQ = QP$ whenever $P, Q \in \mathcal{L}$. A atom of a CSL $\mathcal{L}$ is a non-zero projection $E$ on $\mathcal{H}$ such that for every $L \in \mathcal{L}$, either $E \leq L$ or $EL = 0$. A CSL is called continuous if it has no atoms. A **CSL algebra** is a reflexive algebra $A$ of the form $A = \text{Alg} \mathcal{L}$ for some CSL $\mathcal{L}$; equivalently, CSL algebras are the reflexive operator algebras containing a maximal abelian selfadjoint algebra (masa). CSL’s and CSL algebras were introduced and studied in depth by Arveson in [2].

A **CDCSL algebra** is a CSL algebra $A$ with the property that the Hilbert-Schmidt operators belonging to $A$ are weakly dense in $A$. We note that usually the definition of a CDCSL algebra is given in terms of a strong distributivity property of its subspace lattice; however, the definition given above is equivalent to it [6, Theorem 23.7].

A **nest** is a totally ordered CSL, and a **nest algebra** is an operator algebra $A \subseteq \mathcal{B}(\mathcal{H})$ of the form $A = \text{Alg} \mathcal{L}$ for some nest $\mathcal{L} \subseteq \mathcal{P}(\mathcal{H})$. By
the Erdos Density Theorem [6, Theorem 3.11], the class of CDCSL algebras contains all nest algebras. If \( L \) is a nest, let \( L^\perp = \{ L^\perp : L \in L \} \), where for a projection \( P \) we have set \( P^\perp = I - P \). Then \( \text{Alg} L^\perp = (\text{Alg} L)^* \).

Throughout the paper, \( I \) will denote the interval \([0, 1]\) equipped with the Lebesgue measure, \( H \) will denote the Hilbert space \( L^2(I) \) and \( D \equiv L^\infty(0, 1) \) will denote the corresponding multiplication masa. For each \( t \in [0, 1] \), let \( N_t \) be the projection onto the subspace \( \{ f \in L^2(0, 1) : f(s) = 0, \ \text{a.e.} \ s < t \} \). The nest \( N = \{ N_t : 0 \leq t \leq 1 \} \) is known as the Volterra nest, and the corresponding reflexive algebra \( A_v = \text{Alg} N \) is called the Volterra nest algebra. The von Neumann algebra \( N'' \) generated by \( N \) is equal to \( D \) and is in particular a masa; nests with this property are called multiplicity free.

(Here, and in the sequel, for a subset \( S \subseteq B(H) \) we let \( S' = \{ T \in B(H) : TS = ST, \ \forall S \in S \} \) be the commutant of \( S \).) It is well-known that every continuous multiplicity free nest is unitarily equivalent to the Volterra nest. We refer the reader to [6] for the theory of nest algebras.

We denote by \([T]\) the linear span of a subset \( T \subseteq V \) of a linear space \( V \), and by \( V \otimes W \) the algebraic tensor product of the linear spaces \( V, W \). If \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are Hilbert spaces, we denote by \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) their Hilbertian tensor product. If \( \mathcal{A} \subseteq B(\mathcal{H}_1) \) and \( \mathcal{B} \subseteq B(\mathcal{H}_2) \) we let \( \mathcal{A} \otimes \mathcal{B} \) denote the weakly closed subalgebra of \( B(\mathcal{H}_1 \otimes \mathcal{H}_2) \) generated by the elementary tensors \( A \otimes B \), where \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \). Tensor products of nest algebras were studied in detail in [12] where it was shown that if \( N_i \) is a nest and \( A_i = \text{Alg} N_i \), \( i = 1, \ldots, n \), then \( \mathcal{A} \overset{def}{=} \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \) is a CSL algebra. In fact, \( \mathcal{A} = \text{Alg}(N_1 \otimes \cdots \otimes N_n) \) where, if \( L_1 \) and \( L_2 \) are subspace lattices, \( L_1 \otimes L_2 \) is the smallest subspace lattice containing the projections of the form \( P_1 \otimes P_2 \), where \( P_1 \in L_1 \) and \( P_2 \in L_2 \).

We denote by \( C_2(\mathcal{H}) \) the ideal of all Hilbert-Schmidt operators on a Hilbert space \( \mathcal{H} \), and by \( \| \cdot \|_2 \) the Hilbert-Schmidt norm. Let \( (X, \mu) \) be a standard measure space and \( \mathcal{H} = L^2(X) \). The Hilbert-Schmidt operators on \( \mathcal{H} \) are precisely the integral operators \( T_h \) where, for a function \( h \in L^2(X \times X, \mu \times \mu) \) we let

\[
(T_h f)(y) = \int_X h(x, y) f(x) d\mu(x), \quad f \in \mathcal{H}, y \in X.
\]

The function \( h \) is called the integral kernel of \( T_h \). Moreover, if \( \mathcal{H} = H = L^2(I) \) then \( T_h \in \mathcal{A}_v \) if and only if, up to a null set,

\[
\text{supp } h \subseteq \Delta_v \overset{def}{=} \{(x, y) \in I \times I : x \leq y \},
\]

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where \( \text{supp} \, h = \{(x, y) \in I \times I : h(x, y) \neq 0\} \) is the support of the function \( h \) (defined up to a null set).

The Hilbert space \( H \otimes \cdots \otimes H \) can be naturally identified with \( L^2(I^n) \), where \( I^n = [0, 1] \times \cdots \times [0, 1] \) is equipped with the (\( n \)-dimensional) Lebesgue measure. For \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in I^n \) we write \( x \leq y \) if \( x_i \leq y_i \) for each \( i = 1, \ldots, n \). It follows from [12, Proposition 2.1] that if \( h \in L^2(I^n \times I^n) \) then the Hilbert-Schmidt operator \( T_h \) belongs to the algebra \( \mathcal{A}_n \otimes \cdots \otimes \mathcal{A}_n \) if and only if, up to a null set, \( \text{supp} \, h \subseteq \Delta \) defined as \( \{(x, y) \in I^n \times I^n : x \leq y\} \).

If \( A_i, i = 1, \ldots, n, \) are operator algebras and \( A = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \) we let
\[
N_e(A) = \{ T_1 \otimes \cdots \otimes T_n : T_i \in N(A_i), i = 1, \ldots, n \}.
\]
It is obvious that \( N_e(A) \subseteq N(A) \).

### 3 Proof of the result

In this section we give a proof of our main result, Theorem 1.2. We will need several auxiliary facts.

**Lemma 3.1** (i) Let \( H = H \otimes \cdots \otimes H \) and \( A \in \mathcal{B}(H) \) be a Hilbert-Schmidt operator with integral kernel \( h \in L^2(I^n \times I^n) \). Assume that the set \( \Delta \cap \text{supp} \, h \) has positive measure. Then there exist \( P_i \in \mathcal{N}, i = 1, \ldots, n, \) such that \( (P_1 \otimes \cdots \otimes P_n)A(P_1^\perp \otimes \cdots \otimes P_n^\perp) \neq 0 \).

(ii) Let \( H_i \) be a Hilbert space, \( \mathcal{N}_i \) be a continuous multiplicity free nest on \( H_i, \mathcal{A}_i = \text{Alg} \mathcal{N}_i, i = 1, \ldots, n, \mathcal{H} = H_1 \otimes \cdots \otimes H_n \) and \( \mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \). Then the linear span of
\[
\bigcup \{(P_1 \otimes \cdots \otimes P_n)\mathcal{C}_2(\mathcal{H})(P_1^\perp \otimes \cdots \otimes P_n^\perp) : P_i \in \mathcal{N}_i, i = 1, \ldots, n\}
\]
is dense in \( \mathcal{C}_2(\mathcal{H}) \cap \mathcal{A} \) in the Hilbert-Schmidt norm.
Proof. (i) For an element \( t = (t_1, \ldots, t_n) \in I^n \) write
\[
[t, 1] = \{(s_1, \ldots, s_n) \in I^n : t_i \leq s_i \leq 1, i = 1, \ldots, n\}
\]
and
\[
[0, t] = \{(s_1, \ldots, s_n) \in I^n : 0 \leq s_i < t_i, i = 1, \ldots, n\}.
\]
Let \( \{t_k\}_{k \in \mathbb{N}} \) be a dense subset of \( I^n \), \( L_k = P([t_k, 1]) \) and \( M_k = P([0, t_k]) \), \( k \in \mathbb{N} \). Suppose that \( L_k AM_k = 0 \) for each \( k \in \mathbb{N} \). This implies that \( h\chi_{(0,t_k) \times [t_k,1]} = 0 \) for each \( k \in \mathbb{N} \). Since \( \cup_{k \in \mathbb{N}} [0, t_k) \times [t_k, 1] = \Delta^o \) (where \( \Delta^o \) is the interior of \( \Delta \)) and \( (\mu \times \mu)(\Delta \setminus \Delta^o) = 0 \), it follows that the set \( \Delta \cap \text{supp} \ h \) has measure zero, a contradiction.

(ii) Since each continuous multiplicity free nest is unitarily equivalent to the Volterra nest, we may assume that \( \mathcal{N}_i = \mathcal{N}, i = 1, \ldots, n \). Suppose that \( T \in C_2(\mathcal{H}) \cap \mathcal{A} \) is orthogonal to the spaces of the form \( PC_2(\mathcal{H})Q \) where \( P = P_1 \otimes \cdots \otimes P_n \) and \( Q = P_1^\perp \otimes \cdots \otimes P_n^\perp \) for some \( P_i \in \mathcal{N}, i = 1, \ldots, n \). Thus, \( \text{tr}(PSQT^*) = 0 \) for all \( S \in C_2(\mathcal{H}) \) and \( P \) and \( Q \) of the above form. It follows that \( \text{tr}(SQT^*P) = 0 \) for all such \( S, P \) and \( Q \) and so \( PTQ = 0 \) for all such \( P \) and \( Q \). Suppose that \( h \in L^2(I^n \times I^n) \) is the integral kernel of \( T \). By (i), \( \Delta \cap \text{supp} \ h \) is a null subset of \( I^n \times I^n \). However, since \( T \in \mathcal{A} \), we have that \( \text{supp} \ h \subseteq \Delta \) up to a null set. It follows that \( \text{supp} \ h \) is a null set, and hence \( T = 0 \). \( \diamond \)

Lemma 3.2 Let \( H_i \) be a Hilbert space, \( \mathcal{N}_i \) be a continuous nest on \( H_i \), \( \mathcal{A}_i = \text{Alg}\mathcal{N}_i, i = 1, \ldots, n \), \( \mathcal{H} = H_1 \otimes \cdots \otimes H_n \) and \( \mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \). Then \( [\mathcal{N}_c(\mathcal{A})] \) is weakly dense in \( B(\mathcal{H}) \).

Proof. It is clear that
\[
[\mathcal{N}_c(\mathcal{A})] = [\mathcal{N}(\mathcal{A}_1)] \otimes \cdots \otimes [\mathcal{N}(\mathcal{A}_n)].
\]
Let \( \mathcal{B}_i = [\mathcal{N}(\mathcal{A}_i)]^{\perp \perp} \); since \( \mathcal{N}(\mathcal{A}_i) \) is a selfadjoint semi-group, \( \mathcal{B}_i \) is the \( C^* \)-algebra generated by \( \mathcal{N}(\mathcal{A}_i) \). By Corollary 3.5 (i) of [19], \( \mathcal{B}_i^w = \mathcal{B}(H_i), i = 1, \ldots, n \). Let \( T_i \in \mathcal{B}(H_i) \). By the Kaplansky Density Theorem, there exists a bounded net \( \{T_i^\nu\} \subseteq \mathcal{B}_i \) converging weakly to \( T_i, i = 1, \ldots, n \). We may assume that \( \{\nu_i\} \) coincides with the same directed set \( \{\nu\} \) for all \( i \). It is obvious that
\[
((T_1^\nu \otimes \cdots \otimes T_n^\nu)\xi, \eta) \rightarrow ((T_1 \otimes \cdots \otimes T_n)\xi, \eta)
\]
for all $\xi, \eta \in H_1 \otimes \cdots \otimes H_n$. Since the nets $\{T^\nu_i\}$ are bounded, it follows that

$$T^\nu_1 \otimes \cdots \otimes T^\nu_n \to T_1 \otimes \cdots \otimes T_n$$

weakly. Since each $T^\nu_i$ can be approximated in the norm topology by a bounded sequence of elements of $[N_e(A)]$, we conclude that $T_1 \otimes \cdots \otimes T_n \in [N_e(A)]^w$. Thus,

$$\mathcal{B}(H_1) \otimes \cdots \otimes \mathcal{B}(H_n) \subseteq [N_e(A)]^w.$$

Since $[N_e(A)]^w$ is a von Neumann algebra, we conclude that $[N_e(A)]^w = \mathcal{B}(H)$. ♦

**Lemma 3.3** Let $H_i$ be a Hilbert space, $N_i$ be a continuous multiplicity free nest on $H_i$, $A_i = \text{Alg} N_i$, $i = 1, \ldots, n$, $\mathcal{H} = H_1 \otimes \cdots \otimes H_n$ and $A = A_1 \otimes \cdots \otimes A_n$. Let $A \in C_2(\mathcal{H})$ be a non-zero Hilbert-Schmidt operator. Then

$$[V AW : V, W \in N_e(A)]^2 = C_2(\mathcal{H}).$$

(1)

**Proof.** Without loss of generality, we may assume that $H_i = L^2(I)$ and that $N_i = \mathcal{N}$ is the Volterra nest for each $i = 1, \ldots, n$. Thus, up to unitary equivalence, $\mathcal{H} = L^2(I^n)$. If $\xi \in \mathcal{H}$ let $\bar{\xi} \in \mathcal{H}$ be the function given by $\bar{\xi}(t) = \xi(t)$, $t \in I^n$. Let $\gamma : \mathcal{H} \otimes \mathcal{H} \to C_2(\mathcal{H})$ be the unitary operator given on elementary tensors by $\gamma(\xi \otimes \eta)(\xi_0) = (\xi_0, \bar{\xi})\eta$. Given $T \in \mathcal{B}(\mathcal{H})$ we let $\tilde{T} \in B(L^2(I^n))$ be the operator defined by $\tilde{T}(\xi) = T^*(\bar{\xi})$, $\xi \in \mathcal{H}$. It is easy to verify that $\tilde{S} \tilde{T} = \tilde{T} \tilde{S}$ for all $S, T \in \mathcal{B}(\mathcal{H})$ and that the mapping $T \to \tilde{T}$ takes $N_e(A)$ onto itself.

A straightforward calculation shows that

$$S \gamma(\zeta) T = \gamma((\tilde{T} \otimes S) \zeta), \quad S, T \in \mathcal{B}(\mathcal{H}), \quad \zeta \in \mathcal{H} \otimes \mathcal{H}. \quad (2)$$

Let $\zeta_0 = \gamma^{-1}(A)$. By (2),

$$\gamma^{-1} \left( [V AW : V, W \in N_e(A)]^2 \right) = [(W \otimes V)(\zeta_0) : V, W \in N_e(A)]^2. \quad (3)$$

Denote by $\mathcal{E}$ the right hand side of (3). It follows from the previous paragraph that $\mathcal{E}$ is invariant under each operator of the form $T \otimes S$ where $S, T \in N_e(A)$. Hence, $\mathcal{E}$ is invariant under $[N_e(A)] \otimes [N_e(A)]$. Lemma 3.2 now implies that
\( \mathcal{E} \) is invariant under \( \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \) and since \( \mathcal{E} \neq \{0\} \) we have that \( \mathcal{E} = \mathcal{H} \otimes \mathcal{H} \). \( \Diamond \)

Let \( \mathcal{A}_i \) be a nest algebra acting on a Hilbert space \( H_i, \mathcal{N}_i = \text{Lat} \mathcal{A}_i \) and \( P_i \in \mathcal{N}_i \) where \( i = 1, \ldots, n \). Set \( \mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \), \( P = P_1 \otimes \cdots \otimes P_n \) and

\[
N_P(\mathcal{A}) = \{(V_1 \otimes \cdots \otimes V_n) + (W_1 \otimes \cdots \otimes W_n) : V_i, W_i \in N(\mathcal{A}_i), V_i = P_i V_i P_i, W_i = P_i^\perp W_i P_i^\perp, i = 1, \ldots, n\}.
\]

**Lemma 3.4** Let \( H_i \) be a Hilbert space, \( \mathcal{A}_i \) be a nest algebra acting on \( H_i, i = 1, \ldots, n \), and \( \mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \). For each \( i = 1, \ldots, n \), let \( \mathcal{C}_i \) be either equal to \( \mathcal{A}_i \) or to \( \mathcal{A}_i^* \), and set \( \mathcal{C} = \mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n \). Let \( P_i \in \text{Lat} \mathcal{C}_i, i = 1, \ldots, n \) and \( P = P_1 \otimes \cdots \otimes P_n \). Then \( N_P(\mathcal{C}) \subseteq N(\mathcal{A}) \).

**Proof.** Let \( A = A_1 \otimes \cdots \otimes A_n \) where \( A_i \in \mathcal{A}_i, i = 1, \ldots, n \), and \( T \in N_P(\mathcal{C}) \). Thus, \( T = V + W \) where \( V = V_1 \otimes \cdots \otimes V_n \), \( W = W_1 \otimes \cdots \otimes W_n \) for some \( V_i, W_i \in N(\mathcal{C}_i) \) with \( V_i = P_i V_i P_i \) and \( W_i = P_i^\perp W_i P_i^\perp \), \( i = 1, \ldots, n \). Then \( TAT^* = WAV^* + WAV^* + VAW^* + WAW^* \). Since \( V, W \in N(\mathcal{A}) \), we have that \( WAV^*, VAW^* \in \mathcal{A} \). We will show that \( WAV^*, VAW^* \in \mathcal{A} \).

Note that

\[
WAV^* = (P_1^\perp W_1 P_1^\perp A_1 P_1 V_1^* P_1) \otimes \cdots \otimes (P_n^\perp W_n P_n^\perp A_n P_n V_n^* P_n)
\]

and, for each \( i \), either \( P_i \in \mathcal{N}_i \) or \( P_i \in \mathcal{N}_i^\perp \). If \( P_i \in \mathcal{N}_i \), then \( P_i^\perp A_i P_i = 0 \) and so \( P_i^\perp W_i P_i^\perp A_i P_i V_i^* P_i = 0 \). If \( P_i \in \mathcal{N}_i^\perp \) then

\[
P_i^\perp W_i P_i^\perp A_i P_i V_i^* P_i \in P_i^\perp \mathcal{B}(H_i) P_i \subseteq \mathcal{A}_i.
\]

It follows that \( WAV^* \in \mathcal{A} \). Similar arguments show that \( VAW^* \in \mathcal{A} \). It follows that \( TAT^* \subseteq \mathcal{A} \). Similarly, \( T^*AT \in \mathcal{A} \) and hence \( T \in N(\mathcal{A}) \). \( \Diamond \)

The next proposition shows that the normalisers of the tensor product of finitely many copies of the Volterra nest algebra act transitively on certain Hilbert-Schmidt operators.

**Proposition 3.5** Let \( H_i \) be a Hilbert space, \( \mathcal{N}_i \) be a continuous multiplicity free nest acting on \( H_i, \mathcal{A}_i = \text{Alg} \mathcal{N}_i, i = 1, \ldots, n, \mathcal{H} = H_1 \otimes \cdots \otimes H_n \) and \( \mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \). Suppose that \( A \in \mathcal{A} \) is a non-zero Hilbert-Schmidt operator and \( P_i \in \mathcal{N}_i, i = 1, \ldots, n \), are such that \( A = (P_1 \otimes \cdots \otimes P_n) A (P_1^\perp \otimes \cdots \otimes P_n^\perp) \).

Then

(i) \([(ST)A(ST)^* : S \in N_e(\mathcal{A}), T \in N_P(\mathcal{A})]_{\text{\textendash}1} = \mathcal{C}_2(\mathcal{H}) \cap \mathcal{A}, \) and

(ii) \([(ST)A(ST)^* : S \in N_e(\mathcal{A}), T \in N_P(\mathcal{A})]_{\text{\textendash}w^*} = \mathcal{A}. \)
Proof. (i) Let \( P = P_1 \otimes \cdots \otimes P_n \) and \( Q = P_1^\perp \otimes \cdots \otimes P_n^\perp \). Fix \( V_i, W_i \in N(A_i) \) with \( V_i = P_i V_i P_i \) and \( W_i = P_i^\perp W_i P_i^\perp \), \( i = 1, \ldots, n \), and set \( V = V_1 \otimes \cdots \otimes V_n \), \( W = W_1 \otimes \cdots \otimes W_n \) and \( T = V + W \). Since \( A = PAQ \), we have that

\[
TAT^* = (TP)AQT^* = VAW^*.
\]

By Lemma 1.1 (ii) of [19], for any projection \( E_i \in \mathcal{N}_i \) we have that \( N(E_i A_i |_{E_i H_i}) = EN(A_i)|_{E_i H_i} \). It now follows from (4) that if \( C_i = P_i A_i |_{P_i H_i} \) and \( B_i = P_i^\perp A_i |_{P_i^\perp H_i} \), \( i = 1, \ldots, n \), then

\[
\{ TAT^* : T \in N_P(A) \} = \{ VAW : V = V_1 \otimes \cdots \otimes V_n, W = W_1 \otimes \cdots \otimes W_n, V_i \in N(C_i), W_i \in N(B_i), i = 1, \ldots, n \}.
\]

Here we have identified an operator \( B \) acting on a subspace \( H_1 \) of a Hilbert space \( H_2 \) with the operator \( B \oplus 0 \) acting on \( H_2 \).

The algebras \( C_i, B_i \) are continuous multiplicity free nest algebras, and hence are all unitarily equivalent to \( A_v \). Lemma 3.3 now implies that

\[
PC_2(H)Q \subseteq [TAT^* : T \in N_P(A)]^{\| \cdot \|_2}.
\]

Let \( E_i \) be a non-zero projection in \( \mathcal{N}_i \), \( i = 1, \ldots, n \). Then there exists a unitary operator \( U_i \in N(A_i) \) such that \( U_i P_i U_i^* = E_i \), \( i = 1, \ldots, n \). It follows that, if \( E = E_1 \otimes \cdots \otimes E_n \) and \( F = E_1^\perp \otimes \cdots \otimes E_n^\perp \), then

\[
EC_2(H)F \subseteq [(ST)A(ST)^* : S \in N_e(A), T \in N_P(A)]^{\| \cdot \|_2}.
\]

Lemma 3.1 (ii) implies that

\[
C_2(H) \cap A \subseteq [(ST)A(ST)^* : S \in N_e(A), T \in N_P(A)]^{\| \cdot \|_2}.
\]

The converse inclusion follows from Lemma 3.4 and hence (i) is established.

(ii) follows from (i) and the fact that \( \bar{C_2(H)} \cap \bar{A} = A \) (see [6]). \( \diamond \)

We note the following corollary of Proposition 3.5.

**Corollary 3.6** Let \( A \) be a non-zero Hilbert-Schmidt operator in \( A_v \) and \( P \in \mathcal{N} \) be a projection such that \( A = PAP^\perp \). Then

\[
[TAT^* : T \in N(A_v)]^{\omega^*} = A_v.
\]

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Lemma 3.7 Let $B_i \subseteq B(H)$ be either equal to $A_i$ or to $A_i^*$, $i = 1, \ldots, n$. Set $A = A_1 \otimes \cdots \otimes A_n$ and $B = B_1 \otimes \cdots \otimes B_n$. Then $N(A) = N(B)$ if and only if $A = B$ or $A^* = B$.

Proof. Let first $n = 2$ and $B = A_1 \otimes A_2^*$. We show that $N(A) \neq N(B)$. Let $W \in B(H \otimes H)$ be the unitary operator given by $W(\xi \otimes \eta) = \eta \otimes \xi$, $\xi, \eta \in H$. For $P_1, P_2 \in P(H)$ we have that $W(P_1 H \otimes P_2 H) = P_2 H \otimes P_1 H$. If $A \in A$ and $P_1, P_2 \in \mathcal{N}$ then

$$W^* AW(P_1 H \otimes P_2 H) = W^* A(P_2 H \otimes P_1 H) \subseteq W^*(P_2 H \otimes P_1 H) = P_1 H \otimes P_2 H.$$ 

Thus, $W^* AW \subseteq A$. Since $W = W^*$, we have that $W \in N(A)$.

On the other hand, $W \notin N(B)$. Indeed, let $P_1, P_2$ be any non-trivial projections in $\mathcal{N}$ and $V$ be a partial isometry with initial space $P_1 H$ and final space $P_1^* H$. Then $V \in A_2^*$. However,

$$W^*(I \otimes V)W(P_1 H \otimes P_1^* H) = W^*(I \otimes V)(P_2 H \otimes P_1 H) = W^*(P_2 H \otimes P_1^* H) = P_1 H \otimes P_2^* H,$$

which shows that $W^*(I \otimes V)W \notin B$.

Now suppose that $n$ is arbitrary. If $A \neq B$ and $A^* \neq B$ then there exist finite (possibly empty) tensor products $A_1$ and $A_2$ of Volterra nest algebras such that, up to a permutation of the factors in the tensor products,

$$A = (A_1 \otimes A_2) \otimes A_1 \otimes A_2 \quad \text{and} \quad B = (A_1 \otimes A_2^*) \otimes A_1 \otimes A_2^*.$$ 

Let $T = W \otimes I \otimes I$; the first paragraph of the proof shows that $T \in N(A)$. Assume that $T \in N(B)$. It is easy to see that this implies $W \in N(A_1 \otimes A_2^*)$, a contradiction. \hfill \box

We are now ready to prove our main result.

Proof of Theorem 1.2. As in the proof of Proposition 4.4 of [19], we may assume that $A \cap A^*$ is a masa, that is, that $A = \text{Alg}(\mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n)$ for some continuous multiplicity free nests $\mathcal{N}_1, \ldots, \mathcal{N}_n$ acting on $H_1, \ldots, H_n$, respectively. Suppose that the statement holds if $A_i = A_n$, for each $i = 1, \ldots, n$. Recall that $H = L^2(I)$. There exists a unitary operator $U_i : H \rightarrow H_i$ such that $U_i \mathcal{A}_i U_i^* = \mathcal{A}_i$, $i = 1, \ldots, n$. Let $U = U_1 \otimes \cdots \otimes U_n$. Then $U^* AU = A_0 \overset{\text{def}}{=} A_v \otimes \cdots \otimes A_n$. Let $B_0 = U^* BU$.
It is easy to verify that for any operator algebra \( C \) and a unitary operator \( V \) we have \( N(V^*CV) = V^*N(C)V \). The condition \( N(A) = N(B) \) now implies that \( N(A_0) = N(B_0) \). By the assumption, \( A_0 = B_0 \) or \( A_0 = B_0^* \). It follows that \( A = B \) or \( A = B^* \). Hence, it suffices to establish the statement in the case \( A_i = A_e \) for each \( i = 1, \ldots, n \).

We identify \( \mathcal{H} \) with \( L^2(I^n) \). The condition \( N(A) = N(B) \) easily implies that \( A \cap A^* = B \cap B^* \). For each subset \( G \subseteq \{1, 2, \ldots, n\} \), let

\[
\Delta_G = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in I^n \times I^n : x_i < y_i \text{ if } i \in G, x_i > y_i \text{ if } i \not\in G\}
\]

and \( C_G = C_1 \otimes \cdots \otimes C_n \) where \( C_i = A_e \) if \( i \in G \) and \( C_i = A_i^* \) if \( i \not\in G \). We note that the union \( \cup_{G \subseteq \{1, \ldots, n\}} \Delta_G \) is a subset of \( I^n \times I^n \) of full measure.

Let \( B \in \mathcal{B} \) be a Hilbert-Schmidt operator and \( \kappa \subseteq I^n \times I^n \) be the support of its integral kernel. Since \( (\mu \times \mu)(\kappa) > 0 \) there exists a subset \( G \subseteq \{1, \ldots, n\} \) such that \( (\mu \times \mu)(\Delta_G \cap \kappa) > 0 \). It follows from Lemma 3.1 (i) and the fact that every continuous multiplicity free nest is unitarily equivalent to the Volterra nest that there exist \( P_i \in \text{Lat} C_i, i = 1, \ldots, n \), such that the Hilbert-Schmidt operator

\[
A = (P_1 \otimes \cdots \otimes P_n)B(P_1^\perp \otimes \cdots \otimes P_n^\perp)
\]

is non-zero. Since the diagonals of \( A \) and \( B \) coincide, it follows that \( A \in \mathcal{B} \).

We also have that \( A \in C_G \). Indeed, let \( L_i \in \text{Lat} C_i, i = 1, \ldots, n \). If \( L_i \leq P_i \) for some \( i \) then

\[
(L_1 \otimes \cdots \otimes L_n)(P_1^\perp \otimes \cdots \otimes P_n^\perp) = 0.
\]

If \( P_i \leq L_i \) for each \( i \) then \( P_1 \otimes \cdots \otimes P_n \leq L_1 \otimes \cdots \otimes L_n \) and hence

\[
(P_1 \otimes \cdots \otimes P_n)(L_1 \otimes \cdots \otimes L_n)^\perp = 0.
\]

Thus, in both cases, \( (L_1 \otimes \cdots \otimes L_n)^\perp A(L_1 \otimes \cdots \otimes L_n) = 0 \) and hence \( A \in C_G \).

By Proposition 3.5 (ii),

\[
C_G = \{[ST]A(ST)^* : S \in \mathcal{N}_e(C_G), T \in \mathcal{N}_p(C_G)\}^{-w^*}.
\]

Using the fact that \( \mathcal{N}_e(A) = \mathcal{N}_e(C_G) \) and Lemma 3.4 we conclude that

\[
C_G \subseteq [VAV^* : V \in \mathcal{N}(A)]^{-w^*} = [VAV^* : V \in \mathcal{N}(B)]^{-w^*} \subseteq \mathcal{B}.
\]

Now suppose that there exists a Hilbert-Schmidt operator in \( \mathcal{B} \), say \( T \), and a subset \( F \subseteq \{1, 2, \ldots, n\} \) with \( F \neq G \) such that, if \( \lambda \) is the support of
the integral kernel of $T$, then $(\mu \times \mu)(\Delta_F \cap \lambda) > 0$. The previous paragraph implies that $C_F \subseteq \mathcal{B}$. Suppose that $C_F = E_1 \otimes \cdots \otimes E_n$ where $E_i = \mathcal{A}_i$ if $i \in F$ and $E_i = A_i^*$ if $i \notin F$. Since $F \neq G$, there exists an $i$ such that $C_i = E_i^*$. Let $C \in C_i$ be a non-zero Hilbert-Schmidt operator, $\tilde{C} = C_1 \otimes \cdots \otimes C_n$ where $C_j = I$ if $j \neq i$ and $C_i = C$, and $S = \tilde{C} + \tilde{C}^*$. Then

$$S = S^* \in C_G + C_F \subseteq \mathcal{B}.$$ 

Thus, $S \in \mathcal{A} \cap \mathcal{A}^*$ which implies that $C + C^*$ is a Hilbert-Schmidt operator in $\mathcal{A}_v \cap \mathcal{A}_v^*$. If $C + C^* = 0$ then $C^* = -C \in \mathcal{A}_v$, and hence $C$ is a Hilbert-Schmidt operator such that $C, C^* \in \mathcal{A}_v$. If $h$ is the integral kernel of $C$ then the integral kernel $k$ of $C^*$ is given by $k(x, y) = \overline{h(y, x)}$, $x, y \in I$. It follows that $h$ is supported on $\{(x, x) : x \in I\}$, a set of measure zero, and hence $C = 0$, a contradiction. Thus, $C + C^*$ is a non-zero Hilbert-Schmidt operator in $\mathcal{A}_v \cap \mathcal{A}_v^*$, a contradiction.

We have hence shown that every Hilbert-Schmidt operator in $\mathcal{B}$ has integral kernel supported on $\Delta_G$, and hence belongs to $C_G$. Since $C_2(\mathcal{H}) \cap \mathcal{B}$ is weakly dense in $\mathcal{B}$, we conclude that $\mathcal{B} \subseteq C_G$ and so $\mathcal{B} = C_G$. It follows that $N(C_G) = N(A)$. Lemma 3.7 now implies that either $A = B$ or $A^* = B$. ◇

We finish this note by listing two immediate corollaries of Theorem 1.2. The second of them includes as a special case Theorem 1.1.

**Corollary 3.8** Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A}_i, \mathcal{B}_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, be continuous nest algebras such that $\mathcal{A} = A_1 \otimes \cdots \otimes A_n$ and $\mathcal{B} = B_1 \otimes \cdots \otimes B_m$ are unital subalgebras of $B(\mathcal{H})$. Suppose that $N(A) = N(B)$. Then $n = m$ and, moreover, either $A = B$ or $A^* = B$.

**Corollary 3.9** Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A} \subseteq B(\mathcal{H})$ be a continuous nest algebra and $\mathcal{B} \subseteq B(\mathcal{H})$ be a CDCSL algebra. Suppose that $N(A) = N(B)$. Then either $A = B$ or $A^* = B$.

**Acknowledgement** The authors would like to thank A. Katavolos for his useful remarks on the content of this paper.

**References**


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