C^*-Segal algebras with order unit

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Abstract

We introduce the notion of a (noncommutative) C^*-Segal algebra as a Banach algebra $(A, \|\cdot\|_A)$ which is a dense ideal in a C^*-algebra $(C, \|\cdot\|_C)$, where $\|\cdot\|_A$ is strictly stronger than $\|\cdot\|_C$ on A. Several basic properties are investigated and, with the aid of the theory of multiplier modules, the structure of C^*-Segal algebras with order unit is determined.

Key words: Segal algebra, multiplier module, C^*-Segal algebra, order unitization, σ-unital C^*-algebra

1. Introduction

The concept of a Segal algebra originated in the work of Reiter, cf. [18], on subalgebras of the L^1-algebra of a locally compact group. It was generalized to arbitrary Banach algebras by Burnham in [8]. A *C^*-Segal algebra* is a Banach algebra A which is continuously embedded as a dense, not necessarily self-adjoint ideal in a C^*-algebra. Despite many important examples in analysis, such as the Schatten classes for example, the general structure and properties of C^*-Segal algebras is not well understood. The multiplier algebra and the bidual of self-adjoint C^*-Segal algebras were described in [1, 13] and, in the presence of an approximate identity, the form of the closed ideals of C^*-Segal algebras was given in [6]. Commutative C^*-Segal algebras were studied by Arhippainen and the first-named author in [5].

In this paper, our aim is to develop the basics of a theory of general C^*-Segal algebras, with an emphasis on their order structure. In particular, the

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notion of an order unit turns out to be crucial. However, in contrast to the C^*-algebra case, an order unit of a C^*-Segal algebra cannot serve as a multiplicative identity for the algebra. In fact, it emerges that a C^*-Segal algebra with an order unit cannot even have an approximate identity (bounded or unbounded). This necessitates developing new approaches, since most results on Segal algebras have been obtained under the assumption of an approximate identity. To this end, we will introduce a notion of "approximate ideal" which, together with the theory of multiplier modules, is used to determine the structure of C^*-Segal algebras which either contain an order unit or to which an order unit can be added in a natural way. Among the basic examples of C^*-Segal algebras with order unit are faithful principal ideals of C^*-algebras, which play a prominent role, among others, in the theory of locally compact quantum groups.

Section 2 of this paper is of a preliminary nature. We introduce the fundamental concepts and develop basic properties, some of which have been discussed elsewhere in a narrower context. Our purpose here is to prepare the ground for Section 3 where we devote ourselves to noncommutative C^*-Segal algebras. The main new tools employed are the notion of the approximate ideal (see Definition 2.11) together with the concept of multiplier module (Definition 2.17). Theorem 3.12 contains a characterization of self-adjoint C^*-Segal algebras with an order unit whose norm coincides with the order unit norm. So-called weighted C^*-algebras, which provide the noncommutative analogues of Nachbin algebras, are described in Theorem 3.18; they always possess an order unitization (Proposition 3.22).

2. Irregularity of Banach algebras

In this section, we discuss and analyze Banach algebras which are (possibly non-closed) ideals in a Banach algebra.

2.1. Notation and basic definitions

Throughout this paper, let A be a Banach algebra with norm $\|\cdot\|$. A bimodule M over A, in particular, an ideal of A, is called *faithful* if for each $a \in A \setminus \{0\}$ there are $m, n \in M$ such that $a \cdot m \neq 0$ and $n \cdot a \neq 0$. The Banach algebra A is called *faithful* if it is a faithful bimodule over itself.

The basic notion of this paper is that of the *multiplier seminorm*, defined on A by

$$\|a\|_M := \sup_{\|b\| \leq 1} \{\|ab\|, \|ba\|\} \quad (a, b \in A).$$

It is not difficult to verify that $\|\cdot\|_M$ is an algebra seminorm on A which is a norm if A is faithful. If A has an identity element (denoted by e), then each $a \in A$ satisfies

$$\|a\|_M \leq \|a\| \leq \|e\|\|a\|_M.$$

However, in the non-unital case, the interrelations between $\|\cdot\|$ and $\|\cdot\|_M$ become more involved. The left-hand inequality remains true for every $a \in A$, but even if $\|\cdot\|_M$ is a norm, it need not be equivalent to $\|\cdot\|$.

Example 2.1. Let G be an infinite compact group, and let λ be a Haar measure on it which is normalized such that $\lambda(G) = 1$. For $1 \leq p < \infty$, denote by $L_p(G)$ the Banach space of (equivalence classes of) complex-valued functions f on G such that

$$\|f\|_p := \Big(\int_G |f(t)|^p \, d\lambda(t) \Big)^{\frac{1}{p}} < \infty.$$

With convolution as multiplication, $L_p(G)$ is a Banach algebra. Since every $f, g \in L_p(G)$ satisfy $\|f * g\|_p \leq \|f\|_1 \|g\|_p$, it follows that, for $1 < p < \infty$, the multiplier norm on $L_p(G)$ is not equivalent to $\|\cdot\|_p$. On the other hand, it is well known that the two norms coincide on $L_1(G)$.

In order to simplify the subsequent discussion, we introduce some terminology, following [7, 4].

Definition 2.2. The Banach algebra A is called

(i) *norm regular* if $\|\cdot\|$ and $\|\cdot\|_M$ coincide on A;

(ii) *weakly norm regular* if $\|\cdot\|$ and $\|\cdot\|_M$ are equivalent on A;

(iii) *norm irregular* if $\|\cdot\|$ is strictly stronger than $\|\cdot\|_M$ on A.

Besides the multiplier seminorm, the following family of algebra norms will play a fundamental role in our work. The terminology will be justified shortly.

Definition 2.3. Let $|\cdot|$ be an algebra norm on A. We call it a *Segal norm* if there exist strictly positive constants k and l such that

$$k \, \|a\|_M \leq |a| \leq l \, \|a\|$$

for all $a \in A$.

2.2. Segal algebras

Given a Banach algebra B with norm $\|\cdot\|_B$, recall that A is said to be a *Segal algebra in B* if it is a dense ideal of B and there exists a constant $l > 0$ such that $\|a\|_B \leq l \, \|a\|$ for every $a \in A$. (The second condition is automatically fulfilled if B is semisimple, by [6, Proposition 2.2].) For future reference, we record the following standard result of Barnes [6, Theorem 2.3]. For other basic properties of Segal algebras, see [16, 18].

Lemma 2.4. *Let B be a Banach algebra in which A is a Segal algebra. Then A is a Banach B-bimodule, i.e., there exists a positive constant l such that*

$$\|ax\| \leq l \, \|a\| \|x\|_B \quad \textit{and} \quad \|xa\| \leq l \, \|a\| \|x\|_B$$

for all $a \in A$ and $x \in B$.

In our context, it is natural to reverse the notion of a Segal algebra as follows.

Definition 2.5. By a *Segal extension* of A we mean a pair (B, ι), where

(i) B is a Banach algebra;

(ii) ι is a continuous injective homomorphism from A into B;

(iii) $\iota(A)$ is a dense ideal of B.

Given a Segal extension (B, ι) of A, it is evident that $\iota(A)$ becomes a Segal algebra in B when equipped with the norm $\|\iota(a)\|_\iota := \|a\|$ for $a \in A$. Whenever convenient, we shall regard a Segal extension of A as a Banach algebra in which A is a Segal algebra.

The proposition below establishes a useful relation between Segal extensions of A and Segal norms on A. (Here and in the sequel, we identify a normed algebra with its canonical image in its completion.)

Proposition 2.6. *The following conditions are equivalent for a Banach algebra B:*

(a) *B is a Segal extension of A;*

(b) *B is the completion of A with respect to a Segal norm on A.*

Proof. (a) $\Rightarrow$ (b) It follows from Lemma 2.4 that the multiplier seminorm on A is majorized by the norm on B. Together with the definition of a Segal extension, this means that the restriction of $\|\cdot\|_B$ to A is the desired Segal norm on A.

(b) $\Rightarrow$ (a) It is enough to prove that A is an ideal of B. Let $a \in A$ and $x \in B$. Then there is a sequence (a_n) in A such that $\|a_n - x\|_B \to 0$. Noting that each $b \in A$ satisfies $\|ab\| \le \|a\|\|b\|_M$, it is easy to deduce from the definition of a Segal norm that (aa_n) is a Cauchy sequence in A. Thus, for some $b \in A$, one has $\|b - aa_n\| \to 0$. Now

$$\|b - ax\|_B \le \|b - aa_n\|_B + \|aa_n - ax\|_B \le l\,\|b - aa_n\| + \|a\|_B\|a_n - x\|_B \to 0,$$

where l is some positive constant. Therefore, $ax = b$ and A is a right ideal of B. That A is a left ideal of B is proved in a similar way. □

In particular, this result shows that norm irregular Banach algebras provide the natural framework for our investigation.

Corollary 2.7. *Let A be a faithful Banach algebra. Then the following conditions are equivalent:*

(a) *A is norm irregular;*

(b) *A is a Segal algebra.*

Furthermore, the completion of A under the multiplier norm is a Segal extension of A with the property that any Segal extension of A can be embedded as a dense subalgebra.

Remark 2.8. The assumption that A is faithful is not needed in proving (b) implies (a).

For the remainder of this paper, we shall assume that A is faithful. The normed algebra $(A, \|\cdot\|_M)$ will be denoted by A_M and its completion by $\widetilde{A}_M$. We shall regard every Segal extension of A as a subalgebra of $\widetilde{A}_M$.

2.3. Approximate identities of norm irregular Banach algebras

Given a normed algebra B with norm $\|\cdot\|_B$, recall that an *approximate identity* for B is a net $(e_\alpha)_{\alpha\in\Omega}$ in B such that $\|xe_\alpha - x\|_B \to 0$ and $\|e_\alpha x - x\|_B \to 0$ for every $x \in B$. It is said to be *bounded* if there exists a constant $l > 0$ such that $\|e_\alpha\|_B \leq l$ for all $\alpha \in \Omega$. Moreover, it is said to be *contractive* if $\|e_\alpha\|_B \leq 1$ for all $\alpha \in \Omega$. In case $\Omega = \mathbb{N}$, it is said to be *sequential*.

One of the drawbacks of norm irregular Banach algebras is that they cannot possess a bounded approximate identity. Indeed, it is easy to see that if A has a bounded approximate identity $(e_\alpha)_{\alpha\in\Omega}$, then each $a \in A$ satisfies $\|a\| \leq l\,\|a\|_M$, where $l = \sup_{\alpha\in\Omega} \|e_\alpha\|$. In the context of norm irregular Banach algebras, thus the crucial point turns out to be the existence of a bounded approximate identity with respect to the multiplier norm. In order to make this precise, we first need a simple lemma.

Lemma 2.9. *The following conditions are equivalent:*

(a) *A_M has a bounded approximate identity;*

(b) *$\widetilde{A}_M$ has a bounded approximate identity;*

(c) *A has a Segal extension with a bounded approximate identity.*

The proof is immediate from Proposition 2.6 and the fact that a normed algebra has a bounded approximate identity if and only if its completion has a bounded approximate identity (see, e.g., [10, Lemma 2.1]).

Now, consider the set

$$A\widetilde{A}_M := \{ax : \; a \in A \text{ and } x \in \widetilde{A}_M\}.$$

As A is a Banach $\widetilde{A}_M$-bimodule, we conclude from the Cohen–Hewitt Factorization Theorem [12, Theorem B.7.1] and part (b) of the previous lemma that $A\widetilde{A}_M$ is a closed faithful ideal of A whenever A_M has a bounded approximate identity. Its importance lies in the fact that it is the largest closed ideal of A with an approximate identity (necessarily unbounded in the norm irregular case).

Proposition 2.10. *Let A be a Banach algebra such that A_M has a bounded approximate identity $(e_\alpha)_{\alpha\in\Omega}$. Then:*

(i) *$A\widetilde{A}_M = \widetilde{A}_M A$;*

(ii) *$A\widetilde{A}_M = \{a \in A : \; \|ae_\alpha - a\| \to 0$ and $\|e_\alpha a - a\| \to 0\}$;*

(iii) *$A\widetilde{A}_M$ has an approximate identity;*

(iv) *every closed ideal of A with an approximate identity is contained in $A\widetilde{A}_M$.*

Proof. (i) It is immediate from Lemma 2.4 that the closure of A^2 in A coincides with both $A\widetilde{A}_M$ and $\widetilde{A}_M A$, whence the identity follows.

(ii) In view of (i), the inclusion "$\supseteq$" is evident from the closedness of $A\widetilde{A}_M$ in A, and the inclusion "$\subseteq$" follows easily from Lemmas 2.4 and 2.9(b).

(iii) Noting that the net $(e_\alpha^2)_{\alpha\in\Omega}$ is also a bounded approximate identity for A_M, the assertion follows from (ii).

(iv) Given a closed ideal I of A with an approximate identity, the set I^2 is dense in it, and the statement follows. □

Motivated by this result, we make the following definition.

Definition 2.11. Let A be a Banach algebra such that A_M has a bounded approximate identity. We put $E_A := A\widetilde{A}_M$ and call it the *approximate ideal* of A.

Remark 2.12. Our approach is particularly well suited for the study of Banach algebras having an unbounded approximate identity. That is to say, it is not easy to give an example of a Banach algebra with an approximate identity not bounded in the multiplier norm. In fact, it appears that Willis was the first who constructed such an algebra in [23, Example 5]. Moreover, an application of the Uniform Boundedness Principle yields that if a Banach algebra has a sequential approximate identity, then it is automatically bounded with respect to the multiplier norm; see, for instance, [11, p. 191].

As a consequence of the above discussion, we have the following factorization results.

Corollary 2.13. *Let A be a Banach algebra with an approximate identity bounded under $\|\cdot\|_M$. Then $E_A = A$.*

Corollary 2.14. *Let A be a Banach algebra with a sequential approximate identity. Then $E_A = A$.*

Example 2.15. For $1 \leq p < \infty$, denote by ℓ_p the Banach space of complex-valued sequences $x = (x_n)$ such that

$$\|x\|_p := \Big(\sum_n |x_n|^p\Big)^{\frac{1}{p}} < \infty.$$

Under pointwise multiplication, ℓ_p is a commutative Banach algebra with a sequential approximate identity (e.g., the sequence (e_n), where $e_n(k) = 1$ for every $1 \leq k \leq n$, and $e_n(k) = 0$ for every $k > n$). Furthermore, it is not hard to see that the multiplier norm on ℓ_p coincides with the supremum norm. Together with the preceding corollary and the fact that ℓ_p is dense in the algebra c_0 of complex-valued sequences converging to zero, this yields the well-known factorization property $\ell_p = \ell_p\, c_0$.

We finish this subsection with some useful observations on the approximate ideal.

Lemma 2.16. *Let A be a Banach algebra such that A_M has a bounded approximate identity, and let B be a Segal extension of A with a bounded approximate identity. Then:*

(i) *A^2 is dense in E_A;*

(ii) *$AB = BA = E_A$;*

(iii) *B is a Segal extension of E_A.*

Proof. (i) This was already observed in the proof of Proposition 2.10(i).

(ii) Using the Cohen–Hewitt Factorization Theorem again, one deduces that AB and BA are closed ideals of A. The identities now follow from (i) and the inclusions $A^2 \subseteq AB \subseteq E_A$ and $A^2 \subseteq BA \subseteq E_A$.

(iii) It is enough to prove that E_A is dense in B. But this is immediate from (ii) and the facts that A is dense in B and that $B = B^2$. □

2.4. Multipliers of norm irregular Banach algebras

Multiplier modules will play a central role in this paper, as they allow us to reduce the study of certain properties of A to those of E_A. For a general reference on multiplier modules, see [20, 9].

Definition 2.17. Let A be a Banach algebra such that A_M has a bounded approximate identity, and let B be a Segal extension of A. By a *B-multiplier* of A we mean a pair $m = (m_l, m_r)$ of mappings from B into A such that

$$m_l(xy) = m_l(x)y, \; m_r(xy) = xm_r(y), \text{ and } \; xm_l(y) = m_r(x)y \quad (x, y \in B).$$

Each $a \in A$ determines a B-multiplier (l_a, r_a) of A given by $l_a(x) := ax$ and $r_a(x) := xa$ for $x \in B$. We write $M_B(A)$ for the set of B-multipliers of A. Let $\mathscr{L}(B, A)$ denote the Banach algebra of bounded linear mappings from B into A; this is indeed an algebra because A is a Segal algebra in B. It is routine to verify that $M_B(A)$ is a closed subalgebra of $\mathscr{L}(B, A) \oplus_\infty \mathscr{L}(B, A)^{\mathrm{op}}$. In addition, $M_B(A)$ carries a natural B-bimodule structure defined by

$$x \cdot m := (l_{m_r(x)}, r_{m_r(x)}) \; \text{ and } \; m \cdot x := (l_{m_l(x)}, r_{m_l(x)}) \quad (m \in M_B(A), x \in B).$$

There is a continuous injective algebra and B-bimodule homomorphism $\varphi \colon A \to M_B(A)$ given by $\varphi(a) := (l_a, r_a)$ for $a \in A$. In case B has a bounded approximate identity, the image of E_A under φ is a closed faithful ideal of $M_B(A)$.

Remark 2.18. If A and B coincide, then $M_B(A)$ is just the usual *multiplier algebra* $M(A)$ of A. As mentioned in the Introduction, multiplier algebras of Segal algebras have attracted some attention; see also, e.g., [14, 22]. However, the drawback in the norm irregular case is that, although they can be considered as faithful ideals of $M(A)$, neither E_A nor A is closed in it.

The *strict topology* on $M_B(A)$ is defined by the seminorms

$$m \mapsto \|m_l(x)\| + \|m_r(x)\| \quad (x \in B).$$

We shall require the following lemma, which can be found in [21, Theorem 3.5] and [9, Theorem 2.8], for example. (Regrettably, the concept of a faithful module is confused with *non-degenerate module* in [9]; however, our assumptions in the following straighten any ambiguity out.)

Lemma 2.19. *Let A be a Banach algebra such that A_M has a bounded approximate identity, and let B be a Segal extension of A with a bounded approximate identity. Then:*

(i) *$M_B(A)$ equipped with the strict topology is a complete locally convex algebra;*

(ii) *$\varphi(E_A)$ is strictly dense in $M_B(A)$;*

(iii) *if ϕ is a strictly continuous B-bimodule homomorphism from A into $M(B)$, then it has a unique extension $\widetilde{\phi}$ to a strictly continuous B-bimodule homomorphism of $M_B(A)$ into $M(B)$.*

At the end of this section, we describe a universal property of the multiplier module. Since A is faithful (as an algebra) and dense in B, we conclude that it is a faithful B-bimodule.

Proposition 2.20. *Let A be a Banach algebra such that A_M has a bounded approximate identity. Then, for every Segal extension B of A with a bounded approximate identity, $(M_B(A), \varphi)$ satisfies the following conditions:*

(i) *$M_B(A)$ is a faithful B-bimodule;*

(ii) *$\varphi(E_A) = M_B(A) \cdot B = B \cdot M_B(A)$;*

(iii) *if V is a faithful B-bimodule and ϕ is an injective B-bimodule homomorphism from A into V such that $\phi(E_A) = V \cdot B = B \cdot V$, then there exists a unique injective B-bimodule homomorphism ψ of V into $M_B(A)$ such that $\varphi = \psi \circ \phi$.*

Proof. (i) Straightforward.

(ii) In view of Lemma 2.16(ii), it is sufficient to show that $M_B(A) \cdot B$ is contained in $\varphi(E_A)$. Let $m \in M_B(A)$ and $x \in B$. Then there are $y, z \in B$ such that $x = yz$. One has $m \cdot x = m \cdot yz = (l_{m_l(yz)}, r_{m_l(yz)}) = (l_{m_l(y)z}, r_{m_l(y)z}) \in \varphi(AB) = \varphi(E_A)$, as wanted.

(iii) The desired mapping ψ is given by $\psi(v) := \phi_v$ for $v \in V$, where $\phi_v := (\phi_{l,v}, \phi_{r,v})$ is such that $\phi_{l,v}(x) := \phi^{-1}(v \cdot x)$ and $\phi_{r,v}(x) := \phi^{-1}(x \cdot v)$ for each $x \in B$. $\square$

3. C^*-Segal algebras

In this section, we develop the basics of a theory of Segal algebras in C^*-algebras, with an emphasis on the order structure. In our main results, Theorems 3.12 and 3.18, we characterize C^*-Segal algebras with an order unit.

3.1. General properties of C-Segal algebras*

Definition 3.1. We call A a *C*-Segal algebra* if it has a Segal extension (C, ι), where C is a C^*-algebra. We say A is *self-adjoint* if $\iota(A)$ is closed under the involution of C.

The following lemma shows that the theory developed in the previous section will be applicable in the context of C^*-Segal algebras.

Lemma 3.2. *Let A be a C^*-Segal algebra in the C^*-algebra C. Then there is a positive constant l such that*

$$\|a\|_M \leq l\,\|a\|_C \leq l^2 \|a\|_M$$

for all $a \in A$. Furthermore, A_M has a bounded approximate identity which is contractive under the norm on C.

Proof. Let $l > 0$ be as in Lemma 2.4 and take $a \in A$; then

$$\|a\|_M = \sup_{\|b\| \leq 1} \{\|ab\|, \|ba\|\} \leq \sup_{\|b\| \leq 1} l\,\|b\|\,\|a\|_C = l\,\|a\|_C.$$

From this and the density of A in C it is easy to deduce that $\|\cdot\|_M$ has an extension to an algebra norm $\|\cdot\|'_M \colon C \to \mathbb{R}$ satisfying $\|c\|'_M \leq l\,\|c\|_C$ for all $c \in C$. It follows that

$$\|a\|_C^2 \leq \|a\|_M\,\|a^*\|'_M \leq l\,\|a\|_M \|a^*\|_C = l\,\|a\|_M \|a\|_C,$$

where the first inequality is given by [19, 4.8.4], for instance. Combining the two estimates above yields the desired inequalities. The second statement is [12, Proposition 13.1]. ☐

The next two results describe the ideal structure of C^*-Segal algebras. Here, we may, without loss of generality, assume that $\|\cdot\|_M$ and $\|\cdot\|_C$ coincide on A.

Lemma 3.3. *Let A be a C^*-Segal algebra in the C^*-algebra C. Then the following conditions are equivalent:*

(a) *A has an approximate identity;*

(b) *every closed ideal I of A satisfies $I = A \cap \overline{I}$, where $\overline{I}$ denotes the closure of I in C.*

Proof. For the implication (b) $\Rightarrow$ (a), it is sufficient to show that E_A and A coincide, by Proposition 2.10(iii). Since E_A is a closed ideal of A and C is its Segal extension, see Lemma 2.16(iii), the hypothesis yields that $E_A = A \cap \overline{E_A} = A \cap C = A$, as desired.

For the implication (a) $\Rightarrow$ (b), let I be a closed ideal of A and let $(e_\alpha)_{\alpha \in \Omega}$ be an approximate identity for A. Given $x \in A \cap \overline{I}$ and $\varepsilon > 0$, there exist $\alpha \in \Omega$ and $y \in I$ such that $\|x - xe_\alpha\| < \frac{\varepsilon}{2}$ and $\|x - y\|_C < \frac{\varepsilon}{2\|e_\alpha\|}$. Since $ye_\alpha \in I$ and

$$\|x - ye_\alpha\| \leq \|x - xe_\alpha\| + \|xe_\alpha - ye_\alpha\| \leq \|x - xe_\alpha\| + \|e_\alpha\|\,\|x - y\|_C < \varepsilon,$$

it follows that $A \cap \overline{I}$ is contained in I. The reverse inclusion is trivial. ☐

Proposition 3.4. *Let A be a C*-Segal algebra. For every closed ideal I of A, one has:*

(i) *I is a C*-Segal algebra;*

(ii) *A/I is a C*-Segal algebra whenever A has an approximate identity.*

In particular, I and A/I have an approximate identity whenever A has an approximate identity.

Proof. Let (C, ι) be a Segal extension of A and recall that $\iota(A)$ is a Segal algebra in C with the norm $\|\iota(a)\|_\iota := \|a\|$ for $a \in A$, see Definition 2.5.

Let I be a closed ideal of A, and let J denote the closure of $\iota(I)$ in C. To prove (i), it is enough to show that $\iota(I)$ is an ideal of C because every closed ideal in a C*-algebra is a C*-algebra as well. Given $c \in C$ and $x \in I$, there is a sequence (a_n) in A such that $\|\iota(a_n) - c\|_C \to 0$. Since $a_n x \in I$ for all $n \in \mathbb{N}$ and

$$\|\iota(a_n x) - c\iota(x)\|_\iota = \|\iota(a_n)\iota(x) - c\iota(x)\|_\iota \leq \|x\| \, \|\iota(a_n) - c\|_C \to 0,$$

it follows that $c\iota(x) \in \iota(I)$. In a similar fashion, we see that $\iota(x)c \in I$ wherefore $\iota(I)$ is an ideal of C, as claimed.

Towards (ii), define $\kappa \colon A/I \to C/J$ by $\kappa(a+I) := \iota(a)+J$ for $a \in A$. In view of the commutative diagram below, where the vertical arrows are the quotient mappings, it is routine to verify that κ is a continuous homomorphism with dense image.

$$\begin{array}{ccc} A & \xrightarrow{\iota} & C \\ \downarrow & & \downarrow \\ A/I & \xrightarrow{\kappa} & C/J \end{array}$$

The injectivity of κ is given by Lemma 3.3. Indeed, take $a \in A$ such that $a + I \in \ker \kappa$. Then $\iota(a) \in \iota(A) \cap J = \iota(I)$, so that $a \in I$ and hence $a + I = 0$.

For the last assertion, it suffices to show that I has an approximate identity. Let $(e_\alpha)_{\alpha\in\Omega}$ and $(f_\beta)_{\beta\in\Lambda}$ be approximate identities for A and J, respectively. Given $x \in I$ and $\varepsilon > 0$, we find $\alpha \in \Omega$ and $\beta \in \Lambda$ with $\|x - e_\alpha x\| < \frac{\varepsilon}{2}$ and $\|\iota(x) - \iota(x)f_\beta\|_C < \frac{\varepsilon}{2\,\|e_\alpha\|}$. Since $\iota(e_\alpha x)f_\beta \in \iota(I)J = E_{\iota(I)}$ and

$$\begin{aligned} \|\iota(x) - \iota(e_\alpha x)f_\beta\|_\iota &\leq \|\iota(x) - \iota(e_\alpha x)\|_\iota + \|\iota(e_\alpha x) - \iota(e_\alpha x)f_\beta\|_\iota \\ &\leq \|x - e_\alpha x\| + \|e_\alpha\| \, \|\iota(x) - \iota(x)f_\beta\|_C < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

it follows that $E_{\iota(I)}$ is dense in $\iota(I)$, whence $E_{\iota(I)} = \iota(I)$. □

The following consequence of the above results shows that the constructs of the previous section fit well into the framework of C*-Segal algebras.

Corollary 3.5. *Let A be a C*-Segal algebra. Then E_A and $M_C(A)$ are C*-Segal algebras as well.*

Although a C^*-Segal algebra need not be self-adjoint, at present we have no example of a C^*-Segal algebra whose approximate ideal is not self-adjoint. However, we have the following result.

Proposition 3.6. *Let A be a C^*-Segal algebra. Then the following conditions are equivalent:*

(a) *E_A is self-adjoint;*

(b) *$M_C(A)$ is self-adjoint.*

Moreover, E_A is self-adjoint whenever A is self-adjoint.

Proof. The implication (b) $\Rightarrow$ (a) and the last claim are given by Lemma 2.16(ii) and Proposition 2.20(ii). The implication (a) $\Rightarrow$ (b) follows from the easily verified fact that the image of every multiplier in $M_C(A)$ is contained in E_A. $\square$

3.2. Order structure of C^*-Segal algebras

We now turn our attention to the order structure of C^*-Segal algebras. Let A be a C^*-Segal algebra in the C^*-algebra C. The *positive cone* of A is defined by

$$A_+ := A \cap C_+.$$

Let A_h denote the real vector space of self-adjoint elements of A. Then A_h becomes a partially ordered vector space when equipped with the relation

$$x \leq y \quad \text{if} \quad y - x \in A_+ \quad (x, y \in A_h).$$

An element $u \in A_+$ is called an *order unit* of A if each $x \in A_h$ satisfies $x \leq lu$ for some constant $l > 0$.

Example 3.7. Let X be a locally compact Hausdorff space, and let $v\colon X \to \mathbb{R}$ be an upper semicontinuous function such that $v(t) \geq 1$ for every $t \in X$. Define

$$C_b^v(X) := \{f \in C(X) : \ vf \text{ is bounded on } X\}$$

and

$$C_0^v(X) := \{f \in C(X) : \ vf \text{ vanishes at infinity on } X\},$$

where $C(X)$ denotes the set of all continuous complex-valued functions on X. Equipped with pointwise operations and the *weighted supremum norm*

$$\|f\|_v := \sup_{t \in X} v(t)|f(t)|,$$

$C_b^v(X)$ and $C_0^v(X)$ are self-adjoint C^*-Segal algebras. In fact, they are examples of the so-called Nachbin algebras; see, e.g., [17, 2, 3]. It is easy to see that the function $\frac{1}{v}$ serves as an order unit for $C_b^v(X)$ whenever v is continuous on X.

The following standard lemma is recorded for completeness.

Lemma 3.8. *Let A be a self-adjoint C^*-Segal algebra with an order unit. Then*

$$A = A_h + iA_h \quad \textit{and} \quad A_h = A_+ - A_+.$$

Remark 3.9. Let A be a self-adjoint C^*-Segal algebra. Since the involution is continuous on $(A, \|\cdot\|)$, by [19, 4.1.15], the norm $\|a\|' = \max\{\|a\|, \|a^*\|\}$, $a \in A$ is equivalent to the original norm on A. Thus, replacing $\|\cdot\|$ by $\|\cdot\|'$, we can, and will henceforth, assume without loss of generality that the involution on A is an isometry. Under this hypothesis, we have $\|a\|_C \leq \|a\|$ for each $a \in A$, by [19, 4.1.14], which simplifies the subsequent estimates somewhat.

Every order unit $u \in A$ is *strictly positive*, that is, $\omega(u) > 0$ for every positive functional $\omega \neq 0$ on A. It follows that u is a strictly positive element of the surrounding C^*-algebra C, which is therefore σ-unital; that is, contains a countable contractive approximate identity. E.g., such an approximate identity is given by $u_n = \left(\frac{1}{n} + u\right)^{-1} u$. An immediate consequence of this is the following observation.

Lemma 3.10. *Let A be a self-adjoint C^*-Segal algebra with order unit u. Then, for each $c \in C$, one has $uc = 0$ if and only if $c = 0$.*

The following special C^*-Segal algebras are in the focus of our attention.

Definition 3.11. By an *order unit C^*-Segal algebra* we mean a pair (A, u), where A is a self-adjoint C^*-Segal algebra and u is an order unit of A satisfying

$$\|a\| = \inf\{l > 0 : \ -lu \leq a \leq lu\}$$

for all $a \in A_h$.

We now obtain a characterization of order unit C^*-Segal algebras. In the following, 1 will denote the identity element of $M(C)$.

Theorem 3.12. *Let A be a C^*-Segal algebra in the C^*-algebra C, and let $u \in A_+$ be strictly positive. Put $v = u^{\frac{1}{2}} \in C_+$. Then the following conditions are equivalent:*

(a) *(A, u) is an order unit C^*-Segal algebra;*

(b) *there exists a self-adjoint C-subbimodule D of $M(C)$ containing C and 1 such that $A = vDv$ and $\|vdv\| = \|d\|_C$ for all $d \in D_h$.*

In particular, $E_A = vCv$ and $M_C(A) = vM(C)v$ whenever (A, u) is an order unit C^*-Segal algebra.

For the proof, we need the following result.

Lemma 3.13. *With the assumptions and notation as in Theorem 3.12, let $m \in M(C)_h$. Then, for $l > 0$,*

$$-lu \leq vmv \leq lu \quad \Longleftrightarrow \quad -l1 \leq m \leq l1. \tag{3.1}$$

In particular, $\|m\|_C = \inf\{l > 0 : \ -lu \leq vmv \leq lu\}$ for all $m \in M(C)_h$ and $\|a\|_C = \|vav\|$ for all $a \in A_h$.

Proof. Let $m \in M(C)_h$ and $l > 0$ be such that $-lu \le vmv \le lu$. Then each $c \in C$ satisfies

$$-lc^*uc \le c^*vmvc \le lc^*uc,$$

so that

$$\|(vc)^*m(vc)\|_C \le l\,\|(vc)^*(vc)\|_C.$$

As v is strictly positive, vC is dense in C, and hence $\|c^*mc\|_C \le l\,\|c^*c\|_C$ for all $c \in C$. This implies that $\|m\|_C \le l$ which is the right-hand side in (3.1) above.

To prove the last assertion, first suppose that $m \ge 0$. Then

$$\|m^{\frac{1}{2}}c\|_C^2 = \|c^*mc\|_C \le l\,\|c\|_C^2$$

which directly yields the claim. For arbitrary $m \in M(C)_h$, put $n = \|m\|_C\,1 \pm m$. Since

$$\|c^*nc\|_C \le \big\|\|m\|_C\,c^*c \pm c^*mc\big\|_C \le \big(\|m\|_C + l\big)\|c^*c\|_C$$

we obtain $\|n\|_C \le \|m\|_C + l$ by the above. It follows that

$$0 \le \|m\|_C\,1 \pm m \le \|m\|_C\,1 + l$$

and thus $\|m\|_C \le l$.

Since the right-hand side in (3.1) evidently implies the left-hand side, we find that

$$\inf\{l > 0 :\ -lu \le vmv \le lu\} = \|m\|_C \qquad (m \in M(C)_h);$$

specialising this to $a \in A_h$ and using the definition of the order unit norm we obtain the final assertion. □

Proof of Theorem 3.12. (a) ⇒ (b) Clearly, vCv is a self-adjoint subalgebra of C. By Lemma 3.10, we can define a complete *-algebra norm $\|\cdot\|_v \colon vCv \to \mathbb{R}$ by setting $\|vcv\|_v := \|c\|_C$ for $c \in C$. The remainder of the proof is divided into seven steps.

Step 1. $vCv \subseteq E_A$: Let $c \in C_h$ and set

$$u_n := \big(\frac{1}{n} + u\big)^{-1}u \in C_+ \ \text{ and } \ x_n := vu_ncu_nv \in E_A \quad (n \in \mathbb{N}).$$

Since (u_n) is a contractive approximate identity for C, we can apply Lemma 3.13 to conclude that, for all $n, m \in \mathbb{N}$,

$$\begin{aligned}\|x_n - x_m\| &= \|vu_ncu_nv - vu_mcu_mv\| = \|v(u_ncu_n - u_mcu_m)v\| \\ &= \|u_ncu_n - u_mcu_m\|_C \le \|u_ncu_n - c\|_C + \|c - u_mcu_m\|_C \to 0,\end{aligned}$$

whence (x_n) is a Cauchy sequence in E_A. Since E_A is a closed ideal of A, there exists $x \in E_A$ such that $\|x_n - x\| \to 0$. It follows that

$$\|vcv - x\|_C \le \|vcv - x_n\|_C + \|x_n - x\|_C \le \|u\|_C\,\|c - u_ncu_n\|_C + \|x_n - x\| \to 0$$

(see Remark 3.9). Therefore, $vcv = x$ and so $vC_hv \subseteq E_A$. By the identity $C = C_h + iC_h$, this yields the desired inclusion.

Step 2. *vCv is dense in E_A*: It is sufficient to show that vA^2v is dense in E_A. Since vC and Cv are dense in C, so are vA and Av, by the density of A in C. Let $a, b \in A$ and $\varepsilon > 0$. Then we find $a', b' \in A$ with

$$\|a - va'\|_C < \frac{\varepsilon}{2\,l\,\|b\|} \qquad \text{and} \qquad \|b - b'v\|_C < \frac{\varepsilon}{2\,l\,\|va'\|},$$

where the constant $l > 0$ is as in Lemma 2.4. It follows that

$$\begin{aligned}\|ab - va'b'v\| &\leq \|ab - va'b\| + \|va'b - va'b'v\| \\ &\leq l\,\|b\|\,\|a - va'\|_C + l\,\|va'\|\,\|b - b'v\|_C < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

which, together with the density of A^2 in E_A, see Lemma 2.16(i), proves the claim.

Step 3. *$vCv = E_A$*: In view of the above, it is enough to show that $\|\cdot\|$ and $\|\cdot\|_v$ are equivalent on vCv. By Step 1 and Lemma 3.13, the two norms agree on the self-adjoint part as

$$\|vcv\|_v = \|c\|_C = \|vcv\| \qquad (c \in C_h).$$

The cartesian decomposition of $c \in C$ into its real and imaginary parts thus immediately yields $\|vcv\| \leq 2\,\|vcv\|_v \leq 4\,\|vcv\|$ because both norms are *-norms.

Step 4. *Each $c \in C$ determines unique $c', c'' \in C$ with $cv = vc'$ and $vc = c''v$*: Let $c \in C$. Then $cu, uc \in E_A$, so that $cu = vc'v$ and $uc = vc''v$ for some $c', c'' \in C$, by Step 3. An application of Lemma 3.10 yields the uniqueness of c' and c'' as well as the desired identities. For example,

$$cu = vc'v \quad \Longrightarrow \quad (cv - vc')v = 0 \quad \Longrightarrow \quad (cv - vc')u = 0.$$

The uniqueness of the elements c' and c'' yields

$$cv = vc' = (c')''v \quad \Longrightarrow \quad c = (c')'' \;\; \text{and} \;\; vc = c''v = v(c'')' \quad \Longrightarrow \quad c = (c'')'$$

so that the mappings $c \mapsto c'$ and $c \mapsto c''$ are inverses to each other and define algebra automorphisms on C. Note moreover that $Cv = vC$ and $Cu = uC$ since $c''u = vcv = uc'$.

Step 5. *$M_C(A) = vM(C)v$*: Let $m \in M_C(A)$ and recall that its image is contained in E_A. By Steps 3 and 4 together with Lemma 3.10, one can thus define a pair $s := (s_l, s_r)$ of linear mappings on C by the formulae

$$m_l(c') = vs_l(c)v \;\; \text{and} \;\; m_r(c'') = vs_r(c)v \quad (c \in C).$$

To see that s is a multiplier of C, take $c_1, c_2 \in C$. Then

$$\begin{aligned}vs_l(c_1c_2)v &= m_l((c_1c_2)') = m_l(c_1'c_2') = m_l(c_1')c_2' \\ &= vs_l(c_1)vc_2' = vs_l(c_1)c_2v\end{aligned}$$

and

$$\begin{aligned} vs_r(c_1c_2)v &= m_r((c_1c_2)'') = m_r(c_1''c_2'') = c_1''m_r(c_2'') \\ &= c_1''vs_r(c_2)v = vc_1s_r(c_2)v, \end{aligned}$$

so that $s_l(c_1c_2) = s_l(c_1)c_2$ and $s_r(c_1c_2) = c_1s_r(c_2)$. Moreover,

$$\begin{aligned} vc_1s_l(c_2)v &= c_1''vs_l(c_2)v = c_1''m_l(c_2') \\ &= m_r(c_1'')c_2' = vs_r(c_1)vc_2' \\ &= vs_r(c_1)c_2v, \end{aligned}$$

and therefore $c_1s_l(c_2) = s_r(c_1)c_2$. Consequently, $s \in M(C)$ and $m = vsv$. Indeed, for all $c \in C$,

$$m_l(c)v = m_l(v)c' = vs_l(v)vc' = vs_l(vc)v$$

and

$$vm_r(c) = c''m_r(v) = c''vs_r(v)v = vs_r(cv)v,$$

because $v = v' = v''$. Thus, $m_l = l_v \circ s_l \circ l_v$ and $m_r = r_v \circ s_r \circ r_v$, that is, $m = vsv$. This concludes the proof that $M_C(A) \subseteq vM(C)v$. The reverse inclusion is evident from Steps 3 and 4; thus the identity follows.

Step 6. *$A = vDv$ for some self-adjoint C-subbimodule D of $M(C)$ containing C and* 1: Putting

$$D := \{m \in M(C) : \ vmv \in A\},$$

the statement is clear from Steps 3 and 5 together with the inclusions $E_A \subseteq A \subseteq M_C(A)$.

Step 7. $\|vdv\| = \|d\|_C$ *for all $d \in D_h$*: This is a special case of Lemma 3.13.

(b) $\Rightarrow$ (a) Clearly, A is self-adjoint. To show that u is an order unit of A, let $a \in A_h$. Then $a = vdv$ for some $d \in D_h$. Since

$$-\|d\|_C\, 1 \leq d \leq \|d\|_C\, 1,$$

it follows that

$$-\|d\|_C\, u \leq vdv \leq \|d\|_C\, u,$$

as wanted. The order unit norm property of $\|\cdot\|$ is immediate from the identities

$$\begin{aligned} \|a\| = \|vdv\| = \|d\|_C &= \inf\{l > 0 : \ -l1 \leq d \leq l1\} \\ &= \inf\{l > 0 : \ -lu \leq vdv \leq lu\} \\ &= \inf\{l > 0 : \ -lu \leq a \leq lu\}, \end{aligned}$$

where we have employed Lemma 3.13 again. As a result, (A, u) is an order unit C*-Segal algebra. $\square$

The form $A = vDv$ for a C-subbimodule D of $M(C)$ which the C*-Segal algebra A takes in Theorem 3.12 above is not the natural way one would expect an ideal of C to appear. However, there is a commutation relation hidden in its proof, which we make explicit now.

Corollary 3.14. *Let (A, u) be an order unit C*-Segal algebra in the C*-algebra C and let $v = u^{\frac{1}{2}}$. Then $A = vDv$ for a self-adjoint C-subbimodule D of $M(C)$ and $vC = Cv$. Moreover, $E_A = uC = Cu$ and $M_C(A) = uM(C) = M(C)u$.*

Proof. We shall use the notation of Step 4 in the proof of Theorem 3.12. It was shown there that $vC = Cv$ (which explains why A appears as an ideal in C). It follows immediately that $E_A = vCv = Cu = uC$. To prove the final assertion we extend the automorphism $c \mapsto c'$ and its inverse $c \mapsto c''$ from C to $M(C)$ via $n'c' = (nc)'$, $c'n' = (cn)'$, $n''c'' = (nc)''$ and $c''n'' = (cn)''$ for $n \in M(C)$. For each $c \in C$, we have

$$vnc = (nc)''v = n''c''v = n''vc \quad \text{and} \quad nvc' = ncv = v(nc)' = vn'c'$$

and thus $vn = n''v$ and $nv = vn'$, that is, the identities from Step 4 extend to $M(C)$. As a result, $un' = vnv = n''u$ so that $uM(C) = vM(C)v = M(C)u$ as claimed. □

It follows from the above results that, whenever (A, u) is an order unit C*-Segal algebra in the C*-algebra C, $M_C(A) = uM(C) \subseteq C$ and thus C is a Segal extension of $M_C(A)$.

3.3. Weighted C-algebras*

We now introduce a class of C*-Segal algebras that provide the noncommutative analogue of the Nachbin algebras discussed in Example 3.7 above.

Definition 3.15. By a *weighted C*-algebra* we mean a pair (A, π), where

(i) A is a self-adjoint C*-Segal algebra in the C*-algebra C;

(ii) $\pi\colon A \to M(C)$ is a positive isometric C-bimodule homomorphism.

Remark 3.16. The order structure on the multiplier module $M_C(A)$ is defined such that the positive cone $M_C(A)_+$ agrees with $M_C(A) \cap M(C)_+$.

The link between Nachbin algebras and commutative weighted C*-algebras is given by the result below, proved in [5].

Proposition 3.17. *Let (A, π) be a commutative weighted C*-algebra. Then A is isometrically *-isomorphic to a closed self-adjoint subalgebra of $C_b^v(X)$ for a locally compact Hausdorff space X and a continuous real-valued function v on X with $v(t) \geq 1$ for all $t \in X$. In particular, up to an isometric *-isomorphism, $E_A = C_0^v(X)$ and $M_C(A) = C_b^v(X)$.*

The main result of this subsection establishes a characterization of weighted C*-algebras.

Theorem 3.18. *Let (A, π) be a weighted C^*-algebra. Then there exists $u \in Z(M(C))_+$ such that $A = u\pi(A)$. In particular, $E_A = uC$ and $M_C(A) = uM(C)$.*

Proof. We divide the proof into six steps.

Step 1. $\pi(E_A) = C$: It follows from the assumptions that $\pi(E_A)$ is a closed ideal of C. Since every closed ideal in a C^*-algebra is its own square, one gets

$$\pi(E_A) = \pi(E_A)\pi(E_A) = \pi(E_A\pi(E_A)) = \pi(\pi({E_A}^2)).$$

Since π is injective, it follows that $E_A = \pi({E_A}^2)$. Combining this with the density of E_A in C yields

$$C = \overline{E_A} = \overline{\pi({E_A}^2)} \subseteq \overline{\pi(E_A)} = \pi(E_A),$$

and thus $\pi(E_A) = C$, as claimed.

By Lemma 2.19(iii), π can be extended to a strictly continuous C-bimodule homomorphism $\widetilde{\pi}\colon M_C(A) \to M(C)$.

Step 2. *$\widetilde{\pi}$ is a positive isometric surjective $M(C)$-bimodule homomorphism*: For the $M(C)$-bimodule homomorphism property of $\widetilde{\pi}$, let $m \in M_C(A)$ and $n \in M(C)$. Given $c_1, c_2 \in C$, one has

$$c_1 n\widetilde{\pi}(m)c_2 = \widetilde{\pi}(c_1 nmc_2) = c_1\widetilde{\pi}(nm)c_2,$$

and since C is a faithful ideal of $M(C)$, it follows that $\widetilde{\pi}(nm) = n\widetilde{\pi}(m)$. In a similar way, one obtains that $\widetilde{\pi}(mn) = \widetilde{\pi}(m)n$. To see that $\widetilde{\pi}$ is positive, let $m \in M_C(A)_+$ and $c \in C$. Then c^*mc is in A_+, and so

$$c^*\widetilde{\pi}(m)c = \pi(c^*mc) \geq 0$$

which yields the positivity of $\widetilde{\pi}(m)$. For the isometric property of $\widetilde{\pi}$, let $m \in M_C(A)$. It is routine to check that

$$\|m\| = \|m_l\| = \|m_r\|.$$

Moreover, since each $a \in A$ satisfies

$$\begin{aligned}\|l_a\| = \|r_a\| = \sup_{\|c\|_C \leq 1} \|ac\| = \sup_{\|c\|_C \leq 1} \|\pi(ac)\|_C \\ = \sup_{\|c\|_C \leq 1} \|\pi(a)c\|_C = \|\pi(a)\|_C = \|a\|,\end{aligned}$$

we conclude that

$$\begin{aligned}\|m\| &= \sup_{\|c\|_C \leq 1} \|l_{m_l(c)}\| = \sup_{\|c\|_C \leq 1} \|r_{m_l(c)}\| = \sup_{\|c\|_C \leq 1} \|mc\| \\ &= \sup_{\|c\|_C \leq 1} \|\pi(mc)\|_C = \sup_{\|c\|_C \leq 1} \|\widetilde{\pi}(m)c\|_C = \|\widetilde{\pi}(m)\|_C,\end{aligned}$$

as required. Finally, to establish the surjectivity of $\widetilde{\pi}$, let $n \in M(C)$. By Step 1 and the strict density of C in $M(C)$, there is a net $(m_\alpha)_{\alpha\in\Omega}$ in E_A such that

$$\|nc - \pi(m_\alpha)c\|_C + \|cn - c\pi(m_\alpha)\|_C \to 0$$

for every $c \in C$. Therefore, for all $\alpha, \beta \in \Omega$,

$$\begin{aligned}\|m_\alpha c - m_\beta c\| + \|cm_\alpha - cm_\beta\| &= \|\pi(m_\alpha c - m_\beta c)\|_C + \|\pi(cm_\alpha - cm_\beta)\|_C \\ &= \|\pi(m_\alpha)c - \pi(m_\beta)c\|_C + \|c\pi(m_\alpha) - c\pi(m_\beta)\|_C,\end{aligned}$$

so that $(m_\alpha)_{\alpha\in\Omega}$ is a Cauchy net in the strict topology in $M_C(A)$. Letting $m \in M_C(A)$ be its strict limit, see Lemma 2.19(i), the strict continuity of $\widetilde{\pi}$ entails that $\widetilde{\pi}(m) = n$.

Step 3. $M_C(A) = uM(C)$: By the surjectivity of $\widetilde{\pi}$, there exists $u \in M_C(A)$ such that $\widetilde{\pi}(u)$ is the identity element of $M(C)$. Given $m \in M_C(A)$, one has

$$m = \widetilde{\pi}(u)m = \widetilde{\pi}(um) = u\widetilde{\pi}(m), \tag{3.2}$$

where we have employed the $M(C)$-bimodule homomorphism property of $\widetilde{\pi}$. As a result, $M_C(A) = u\widetilde{\pi}(M_C(A)) = uM(C)$.

Step 4. $A = u\pi(A)$: This is a special case of (3.2).

Step 5. $E_A = uC$: Using Steps 1 and 4, we find

$$E_A = AC = u\pi(A)C = u\pi(E_A) = uC,$$

as claimed.

Step 6. *u belongs to $Z(M(C))_+$*: The centrality of u is immediate from the $M(C)$-bimodule homomorphism property of $\widetilde{\pi}$. To see that it is positive, let $c \in C$. Then

$$c^*u^*c = c^*\widetilde{\pi}(u)u^*c = \pi(c^*uu^*c) = \pi((c^*u)(c^*u)^*) \geq 0,$$

implying that u^*, and therefore u as well, is positive. ☐

Corollary 3.19. *The following properties hold for a weighted C*-algebra (A, π):*

(i) *E_A and $M_C(A)$ are self-adjoint;*

(ii) *$\widetilde{\pi}$ is self-adjoint;*

(iii) *$\|\cdot\|_M$ and $\|\cdot\|_C$ coincide on A.*

Proof. (i) Evident.

(ii) Let $m \in M_C(A)$; then

$$u\widetilde{\pi}(m^*) = m^* = (u\widetilde{\pi}(m))^* = \widetilde{\pi}(m)^*u = u\widetilde{\pi}(m)^*,$$

where we have used (3.2). By Lemma 3.10, it follows that $\widetilde{\pi}(m^*) = \widetilde{\pi}(m)^*$, as wanted.

(iii) In view of Lemma 3.2, it is sufficient to show that A is a contractive bimodule over C, that is,

$$\|ac\| \le \|a\|\|c\|_C \quad \text{and} \quad \|ca\| \le \|a\|\|c\|_C$$

for all $a \in A$ and $c \in C$. But this is immediate from the assumptions on π. $\square$

The next result establishes the uniqueness of the "weight" of a weighted C*-algebra.

Corollary 3.20. *Let A be a self-adjoint C*-Segal algebra in the C*-algebra C. Suppose that $\pi_1, \pi_2 \colon A \to M(C)$ are such that (A, π_1) and (A, π_2) are weighted C*-algebras. Then $\pi_1 = \pi_2$.*

Proof. Let u_1 and u_2 denote positive elements of $M_C(A)$ for which $\widetilde{\pi}_1(u_1) = \widetilde{\pi}_2(u_2) = 1$. Since each $a \in A$ satisfies $a = u_1\pi_1(a) = u_2\pi_2(a)$, we conclude from Lemma 3.10 that $\pi_1 = \pi_2$ if and only if $u_1 = u_2$. To show the latter identity, let $c \in C$. Then

$$\|u_1c\| = \|\pi_1(u_1c)\|_C = \|\widetilde{\pi}_1(u_1)c\|_C = \|c\|_C = \|\widetilde{\pi}_2(u_2)c\|_C = \|\pi_2(u_2c)\|_C = \|u_2c\|$$

which, together with Corollary 3.19(iii), yields

$$\|u_1c\|_C = \|u_1c\|_M = \sup_{\|a\|\le 1} \|u_1ca\| = \sup_{\|a\|\le 1} \|u_2ca\| = \|u_2c\|_M = \|u_2c\|_C$$

for all $c \in C$. It follows that $\|u_1n\|_C = \|u_2n\|_C$ for every $n \in M(C)$ which, as is well known, implies that $u_1 = u_2$ since both are positive elements; see, e.g., [15, Lemma 3.4]. $\square$

Theorems 3.12 and 3.18 suggest that there is a relation between weighted C*-algebras and order unit C*-Segal algebras. In order to make this precise, we need to generalize the notion of a unitization of a C*-algebra.

Definition 3.21. By an *order unitization* of a self-adjoint C*-Segal algebra A we mean a pair (B, ϕ), where

(i) B is an order unit C*-Segal algebra;

(ii) ϕ is a positive isometric homomorphism from A into B;

(iii) $\phi(A)$ is a faithful ideal of B.

In the proposition below, φ denotes the embedding of A into $M_C(A)$, as given in Definition 2.17. The notation will be that of Theorem 3.18.

Proposition 3.22. *Every weighted C*-algebra has an order unitization.*

Proof. Let (A, π) be a weighted C*-algebra. Then $(M_C(A), \varphi)$ is an order unitization of it. Indeed, since $M_C(A) = uM(C)$ and each $n \in M(C)_h$ satisfies

$$\|un\| = \|\widetilde{\pi}(un)\|_C = \|\widetilde{\pi}(u)n\|_C = \|n\|_C,$$

the centrality of u together with Theorem 3.12(b) imply that $M_C(A)$ is an order unit C^*-Segal algebra. The isometric property of φ was established in Step 2 of the proof of Theorem 3.18, and the other required properties are trivial. The proof is complete. $\square$

Among the basic examples of weighted C^*-algebras are the following principal ideals of C^*-algebras.

Proposition 3.23. *Let B be a C^*-algebra, and let $u \in Z(M(B))_+$ be such that uB is faithful in B. Then there is a norm on uB making it into a weighted C^*-algebra, and uB has an approximate identity if and only if it is dense in B.*

Proof. We may assume that $\|u\|_B = 1$. Put $I := uB$ and define $\|\cdot\|_u : I \to \mathbb{R}$ by setting $\|ux\|_u := \|x\|_B$ for $x \in B$. Clearly, $\|\cdot\|_u$ is a norm on I making it into a self-adjoint C^*-Segal algebra in the C^*-algebra J, the closure of I in B. It is not hard to verify that the mapping $ux \mapsto (l_x, r_x)$ is a positive isometric J-bimodule homomorphism from I into $M(J)$. As a result, I is a weighted C^*-algebra. The second statement follows from the identities $E_I = IJ = uBJ = uJ$. $\square$

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