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When is the second local multiplier algebra of a C^*-algebra equal to the first?

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Abstract

We discuss necessary as well as sufficient conditions for the second iterated local multiplier algebra of a separable C^*-algebra to agree with the first.

1. *Introduction*

After the first example of a C^*-algebra A with the property that the second local multiplier algebra $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ of A differs from its first, $M_{\mathrm{loc}}(A)$, was found in [**3**]—thus answering a question first raised in [**17**]—, the behaviour of higher local multiplier algebras began to attract some attention; see, e.g., [**4**], [**7**], [**8**]. That the local multiplier algebra can have a somewhat complicated structure was already exhibited in [**1**], where an example of a non-simple unital C^*-algebra A was given such that $M_{\mathrm{loc}}(A)$ is simple (and hence, evidently, $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$ in this case).

It was proved in [**21**] that, if A is a separable unital C^*-algebra, $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$, provided the primitive ideal space $\mathrm{Prim}(A)$ contains a dense G_δ subset of closed points. One of our goals here is to see how this result can be obtained in a straightforward manner using the techniques developed in [**5**]. The key to our argument is the following observation. Every element in $M_{\mathrm{loc}}(A)$ can be realised as a bounded continuous section, defined on a dense G_δ subset of $\mathrm{Prim}(A)$, with values in the upper semicontinuous C^*-bundle canonically associated to the multiplier sheaf of A. The second local multiplier algebra $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ is contained in the injective envelope $I(A)$ of A, cf. [**12**], [**4**], and every element of $I(A)$ has a similar description as a continuous section of a C^*-bundle corresponding to the injective envelope sheaf of A. To show that $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \subseteq M_{\mathrm{loc}}(A)$ it thus suffices to relate sections of these bundles in an appropriate way. In fact, we shall obtain a more general result in Section 4 which, in particular, unifies the commutative and the unital case. The notion of a quasicentral C^*-algebra, first studied by Delaroche [**9**], [**10**], turns out to be crucial.

It emerges, however, that the short answer to the long question in this paper's title is: *rarely*. In Section 3, we provide a systematic approach to producing separable C^*-algebras with the property that their second local multiplier algebra contains the first as a proper C^*-subalgebra. We obtain a quick proof of Somerset's result [**21**] that $M_{\mathrm{loc}}^{(2)}(A) = M_{\mathrm{loc}}^{(3)}(A)$ for a separable C^*-algebra A which has a dense G_δ subset of closed points in its primitive spectrum in Theorem 3.2 below. In our approach, the injective envelope is employed as a 'universe' in which all C^*-algebras considered are contained as C^*-subalgebras. However, in contrast to previous studies, we do not need additional information on the injective envelope itself.

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In the following we will focus on separable C^*-algebras for a variety of reasons. One of them is the non-commutative Tietze extension theorem, another one the need for a strictly positive element in the bounded central closure of the C^*-algebra. Moreover, just as in Somerset's paper [**21**], Polish spaces (in the primitive spectrum) will play a decisive role. Sections 2 and 3 are fairly self-contained, while Section 4 relies on the sheaf theory developed in [**5**].

2. *Preliminaries*

For a C^*-algebra A, we denote by $\mathrm{Prim}(A)$ its primitive ideal space (with the Jacobson topology); this is second countable if A is separable. For an open subset $U \subseteq \mathrm{Prim}(A)$, let $A(U)$ stand for the closed ideal of A corresponding to U. (Hence, $t \in U$ if and only if $A(U) \nsubseteq t$.) We denote by $\mathcal{D}$ and $\mathcal{T}$ the sets of dense open subsets and dense G_δ subsets of $\mathrm{Prim}(A)$, respectively, and consider them directed under reverse inclusion. The local multiplier $M_{\mathrm{loc}}(A)$ is defined by $M_{\mathrm{loc}}(A) = \varinjlim_{U\in\mathcal{D}} M(A(U))$, where, for $U, V \in \mathcal{D}$ with $V \subseteq U$, the injective *-homomorphism $M(A(U)) \to M(A(V))$ is given by restriction. We put $Z = Z(M_{\mathrm{loc}}(A))$, the centre of $M_{\mathrm{loc}}(A)$. For more details on, and properties of, $M_{\mathrm{loc}}(A)$, we refer to [**2**].

A point $t \in \mathrm{Prim}(A)$ is said to be *separated* if t and every point $t' \in \mathrm{Prim}(A)$ which is not in the closure of $\{t\}$ can be separated by disjoint neighbourhoods. Let $\mathrm{Sep}(A)$ be the set of all separated points of a C^*-algebra A. If A is separable then $\mathrm{Sep}(A)$ is a dense G_δ subset of $\mathrm{Prim}(A)$ [**11**, Théoreme 19].

The following result is useful when computing the norm of a (local) multiplier.

Lemma 2.1. *Let A be a separable C^*-algebra, and let $T \subseteq \mathrm{Sep}(A)$ be a dense G_δ subset. For a countable family $\{f_n \mid n \in \mathbb{N}\}$ of bounded lower semicontinuous real-valued functions on T there exists a dense G_δ subset $T' \subseteq T$ such that $f_{n|T'}$ is continuous for each $n \in \mathbb{N}$.*

This is an immediate consequence of the following well-known facts: $\mathrm{Sep}(A)$ is a Polish space (that is, homeomorphic to a separable, complete metric space) by [**11**, Corollaire 20] and hence any G_δ subset of $\mathrm{Sep}(A)$ is a Polish space [**18**, 4.2.2]; every Polish space is a Baire space [**18**, 4.2.5]; any bounded Borel function into $\mathbb{R}$ defined on a Polish space can be restricted to a continuous function on some dense G_δ subset of the domain [**15**, Sect. 32.II].

In [**21**], p. 322, Somerset introduces an interesting C^*-subalgebra of $M_{\mathrm{loc}}(A)$, which we will denote by K_A: K_A is the closure of the set of all elements of the form $\sum_{n\in\mathbb{N}} a_n z_n$, where $\{a_n\} \subseteq A$ is a bounded family and $\{z_n\} \subseteq Z$ consists of mutually orthogonal projections. (These infinite sums exist in $M_{\mathrm{loc}}(A)$ by [**2**, Lemma 3.3.6], for example. Note also that Z is countably decomposable since A is separable.) It is easy to see that, if the above family $\{a_n\}$ is chosen from K_A instead of A, then the sum $\sum_{n\in\mathbb{N}} a_n z_n$ still belongs to K_A ([**21**, Lemma 2.5]).

The significance of the C^*-subalgebra K_A is explained by the following result. Let $\mathscr{I}_{\mathrm{ce}}(A)$ denote the set of all closed essential ideals of a C^*-algebra A. We denote by $M^{(n)}_{\mathrm{loc}}(A) = M_{\mathrm{loc}}\big(M^{(n-1)}_{\mathrm{loc}}(A)\big)$, $n \geq 2$ the n-fold iterated local multiplier algebra of A.

Lemma 2.2. *Let A be a C^*-algebra such that $K_A \in \mathscr{I}_{\mathrm{ce}}(M_{\mathrm{loc}}(A))$.*
(i) *If $K_I = K_A$ for all $I \in \mathscr{I}_{\mathrm{ce}}(A)$ then $M_{\mathrm{loc}}(K_A) = M(K_A)$.*
(ii) *If $M_{\mathrm{loc}}(K_A) = M(K_A)$ then $M^{(n+1)}_{\mathrm{loc}}(A) = M^{(n)}_{\mathrm{loc}}(A)$ for all $n \geq 2$.*

Proof. Let $J \in \mathscr{I}_{\mathrm{ce}}(K_A)$; then $M(K_A) \subseteq M(J)$. Let $I = J \cap A$; then $I \in \mathscr{I}_{\mathrm{ce}}(A)$ by [**2**, Lemma 2.3.2]. By assumption, we therefore have $K_I = K_A$. Let $m \in M(J)$. As $mI \subseteq K_A$, whenever $\{x_n\}$ is a bounded family in I and $\{z_n\}$ is a family of mutually orthogonal projections in Z, we obtain

$$m\big(\sum_n x_n z_n\big) = \sum_n m x_n z_n \in K_A$$

entailing that $mK_A = mK_I \subseteq K_A$, that is, $m \in M(K_A)$. Consequently, $M(J) \subseteq M(K_A)$ which implies (i).

Towards (ii) observe that $M(K_A) = M_{\rm loc}(K_A) = M_{\rm loc}(M_{\rm loc}(A))$ by hypothesis. Let $J \in \mathscr{I}_{\rm ce}(M^{(2)}_{\rm loc}(A))$. Then $J \cap K_A \in \mathscr{I}_{\rm ce}(M_{\rm loc}(A))$ and, since $J \in \mathscr{I}_{\rm ce}(M(K_A))$, $J \cap K_A \in \mathscr{I}_{\rm ce}(J)$. As a result,

$$M(J) \subseteq M(J \cap K_A) \subseteq M_{\rm loc}(M_{\rm loc}(A)) = M(K_A)$$

and the reverse inclusion $M(K_A) \subseteq M(J)$ is obvious. We conclude that $M^{(3)}_{\rm loc}(A) = M(K_A) = M^{(2)}_{\rm loc}(A)$ which entails the result. □

The next result tells us how to detect multipliers of K_A inside $I(A)$.

LEMMA 2.3. *Let A be a separable C^*-algebra and let $y \in I(A)$. If $ya \in K_A$ for all $a \in A$ then $y \in M(K_A)$.*

Proof. It suffices to show that $y \sum_{n=1}^\infty z_n a_n = \sum_{n=1}^\infty z_n y a_n$ whenever $\{a_n \mid n \in \mathbb{N}\} \subseteq A$ is a bounded family and $\{z_n \mid n \in \mathbb{N}\} \subseteq Z$ consists of mutually orthogonal projections, by [**21**, Lemma 2.5]. Without loss of generality we can assume that $\sum_{n=1}^\infty z_n = 1$.

Putting $y' = y \sum_{n=1}^\infty z_n a_n \in I(A)$ we observe that

$$z_j y' = y z_j \sum_{n=1}^{\infty} z_n a_n = y z_j a_j = z_j y a_j \in K_A$$

by hypothesis. It is therefore enough to prove that, if $y' \in I(A)$ and $y' z_j \in K_A$ for all $j \in \mathbb{N}$, where $\{z_j \mid j \in \mathbb{N}\} \subseteq Z$ consists of mutually orthogonal projections with $\sum_{j=1}^\infty z_j = 1$, then $y' = \sum_{j=1}^\infty y' z_j$, where the latter is computed in K_A.

The assumption $y' z_j \in K_A$ for all $j \in \mathbb{N}$ enables us to write $\sum_{j=1}^\infty y' z_j = \sum_{i=1}^\infty w_i a_i$ for some bounded sequence $(a_i)_{i \in \mathbb{N}}$ in A and a sequence $(w_i)_{i \in \mathbb{N}}$ consisting of mutually orthogonal central projections with $\sum_{i=1}^\infty w_i = 1$. For each $n \in \mathbb{N}$,

$$(w_1 + \ldots + w_n) y' = \sum_{i=1}^{n} w_i a_i.$$

Each projection w_i comes with a closed ideal $I_i = w_i M_{\rm loc}(A) \cap A$ and the C^*-direct sum $I = \bigoplus_{i=1}^\infty I_i$ is a closed essential ideal of A. For $x_i \in I_i$, $1 \le i \le n$, we have

$$\begin{aligned}\big(y' - \sum_{i=1}^{\infty} w_i a_i\big)(x_1 + \ldots + x_n) &= \big(y' - \sum_{i=1}^{\infty} w_i a_i\big)(w_1 + \ldots + w_n)(x_1 + \ldots + x_n)\\ &= \Big(\sum_{i=1}^{n} w_i a_i - \sum_{i=1}^{n} w_i a_i\Big)(x_1 + \ldots + x_n) = 0.\end{aligned}$$

Therefore $\big(y' - \sum_{i=1}^\infty w_i a_i\big)x = 0$ for all $x \in I$ which implies that $y' = \sum_{i=1}^\infty w_i a_i$ by [**4**, Proposition 2.12]. □

Recall that the *bounded central closure*, ${}^{c}A$, of a C^*-algebra A is the C^*-subalgebra $\overline{AZ}$ of $M_{\rm loc}(A)$ [**2**, Section 3.2]. If A is separable then ${}^{c}A$ is σ-unital, which will be useful in Section 3.

In Section 4, we shall need the following auxiliary result whose proof is analogous to the one of [**21**, Lemma 2.2] but we include it here for completeness.

LEMMA 2.4. *Let A be a separable C*-algebra, B a C*-subalgebra of $M_{\mathrm{loc}}(A)$ containing A, and J a closed essential ideal of B. There is $h \in J$ such that $hz \neq 0$ for each non-zero projection $z \in Z$.*

Proof. By [**4**, Proposition 2.14], $I(A) = I(B) = I(M_{\mathrm{loc}}(A))$ and thus $Z(M_{\mathrm{loc}}(B)) = Z$ by [**4**, Theorem 4.12]. For $x \in M_{\mathrm{loc}}(A)$, let $\mathrm{c}(x)$ denote the central support of x, see [**2**], page 52 and Remark 3.3.3. Let $\{h_i\}$ be a maximal family of norm-one elements $h_i \in J$ such that their central supports $\mathrm{c}(h_i)$ are mutually orthogonal. Since A is separable, Z is countably decomposable, hence we may enumerate the non-zero central supports as $\mathrm{c}(h_n)$, $n \in \mathbb{N}$. Put $h = \sum_{n=1}^{\infty} 2^{-n} h_n \in J$. As J is essential, for a non-zero projection $z \in Z$, there is $h' \in J$ with $h'z \neq 0$. If $hz = 0$ then $\mathrm{c}(h)z = 0$ and hence $\mathrm{c}(h_n)z = 0$ for all $n \in \mathbb{N}$. It follows that $\mathrm{c}(h_n)\mathrm{c}(h'z) \leq \mathrm{c}(h_n)z = 0$ which would lead to a contradiction to the maximality assumption on $\{h_n\}$. As a result, $hz \neq 0$ for every non-zero projection $z \in Z$. □

3. *The second local multiplier algebra*

In this section we discuss some necessary and some sufficient conditions for the first and the second local multiplier algebra of a separable C*-algebra A to coincide. The general strategy is that this cannot happen if and only if $M_{\mathrm{loc}}(A)$ contains an essential ideal K with the property that $M(K) \setminus M_{\mathrm{loc}}(A) \neq \emptyset$.

The following proposition introduces the decisive topological condition in $\mathrm{Prim}(A)$.

PROPOSITION 3.1. *Let A be a separable C*-algebra such that* $\mathrm{Prim}(A)$ *contains a dense G_δ subset consisting of closed points. Then K_A is an essential ideal in $M_{\mathrm{loc}}(A)$.*

Proof. Since K_A is a C*-subalgebra of $M_{\mathrm{loc}}(A)$, it suffices to show that, whenever m is a multiplier of a closed essential ideal of A and $a \in K_A$, $ma \in K_A$; in fact, we can assume that $a \in A$, by Lemma 2.3.

Let $U \subseteq \mathrm{Prim}(A)$ be a dense open subset and take $m \in M(A(U))$. For $t \in U$, let $\tilde{t} \in \mathrm{Prim}(M(A(U)))$ denote the corresponding primitive ideal under the canonical identification of $\mathrm{Prim}(A)$ with an open dense subset of $\mathrm{Prim}(M(A(U)))$. Let $\{b_n \mid n \in \mathbb{N}\}$ be a countable dense subset of A, and let T be the dense G_δ subset $T = \mathrm{Sep}(A) \cap U$. Note that, by Lemma 2.1, there is a dense G_δ subset $T' \subseteq T$ such that $t \mapsto \|(m - b_n)a + \tilde{t}\|$ is continuous for all $n \in \mathbb{N}$ when restricted to T'.

Let $\varepsilon > 0$ and take $t \in T'$. Since A is separable and t is a closed point, the canonical mapping $M(A(U)) \to M(A/t)$ is surjective [**18**, 3.12.10] and, denoting by $\tilde{m}$ the image of m under this mapping, we have $(m - b_n)a + \tilde{t} = (\tilde{m} - (b_n + t))(a + t)$. As $\{b_n + t \mid n \in \mathbb{N}\}$ is dense in A/t and A/t is strictly dense in its multiplier algebra, there is b_k such that $\|(\tilde{m} - (b_k + t))(a + t)\| < \varepsilon$. By the above-mentioned continuity there is therefore an open subset $V \subseteq \mathrm{Prim}(A)$ containing t such that

$$\|(m - b_k)a + \tilde{s}\| < \varepsilon \qquad (s \in V \cap T').$$

Letting $z = z_V \in Z$ be the projection from $A(V) + A(V)^\perp$ onto $A(V)$ we conclude that $\|zma - zb_k a\| = \sup_{s \in V \cap T'} \|(m - b_k)a + \tilde{s}\| \leq \varepsilon$.

We now choose a (necessarily countable) maximal family $\{z_k\} \subseteq Z$ of mutually orthogonal projections such that $\|z_k ma - z_k b_k a\| \le \varepsilon$ for each k. Then $\sup z_k = 1$ and $\big\|\sum_k (z_k ma - z_k b_k a)\big\| \le \varepsilon$. As $ma = \sum_k z_k ma$ and $\sum_k z_k b_k a \in K_A$ we conclude that $ma \in K_A$ as claimed proving that K_A is an ideal in $M_{\rm loc}(A)$.

In order to show that K_A is essential let $y \in M_{\rm loc}(A)$ be such that $yK_A = 0$. Then, in particular, $yA = 0$ and thus $y = 0$ by [**2**, Proposition 2.3.3]. □

The next result was first obtained in [**21**, Theorem 2.7] but we believe our approach is more direct and more conceptual.

THEOREM 3.2. *Let A be a separable C*-algebra such that* Prim(A) *contains a dense G_δ subset consisting of closed points. Then $M_{\rm loc}^{(3)}(A) = M_{\rm loc}^{(2)}(A)$ and coincides with $M(K_A)$.*

Proof. Combining Proposition 3.1 with Lemma 2.2 all we need to show is that $K_I = K_A$ for each $I \in \mathscr{I}_{\rm ce}(A)$. Taking $I \in \mathscr{I}_{\rm ce}(A)$, the inclusion $K_I \subseteq K_A$ is evident. Let $U \subseteq \mathrm{Prim}(A)$ be the open dense subset such that $I = A(U)$. Let $T \subseteq \mathrm{Prim}(A)$ be a dense G_δ subset consisting of closed and separated points. Fix $a \in A$ and let $\varepsilon > 0$. For $t \in U \cap T$, $(I + t)/t = A/t$ as t is a closed point. Therefore there is $y \in I$ such that $y + t = a + t$ and hence $N(a - y)(t) = 0$. The continuity of the norm function at t ([**5**, Lemma 6.4]) yields an open neighbourhood V of t such that $N(a - y)(s) < \varepsilon$ for all $s \in V$. Letting $z = z_V \in Z$ be the projection corresponding to V we obtain $\|z(a - y)\| \le \varepsilon$. The same maximality argument as in the proof of Proposition 3.1 provides us with a family $\{z_k\}$ of mutually orthogonal projections in Z and a bounded family $\{y_k\}$ in I with the property that $\big\|a - \sum_k y_k z_k\big\| \le \varepsilon$. This shows that $A \subseteq K_I$ and as a result $K_A \subseteq K_I$ as claimed. □

It was shown in [**7**], see also [**4**, Section 6], that the C*-algebra $A = C[0,1] \otimes K(H)$, where $H = \ell^2$, has the property that $M_{\rm loc}(A) \ne M_{\rm loc}(M_{\rm loc}(A))$. In the following result, we explore a sufficient condition on the primitive ideal space that guarantees this phenomenon to happen.

We shall make use of some topological concepts. Recall that a topological space X is called *perfect* if it does not contain any isolated points. If the closure of each open subset of X is open, then X is said to be *extremally disconnected*. Thus, X is not extremally disconnected if and only if it contains an open subset which has non-empty boundary. It is a known fact that an extremally disconnected metric space must be discrete.

THEOREM 3.3. *Let X be a perfect, second countable, locally compact Hausdorff space. Let $A = C_0(X) \otimes B$ for some non-unital separable simple C*-algebra B. Then $M_{\rm loc}(A) \ne M_{\rm loc}(M_{\rm loc}(A))$.*

Proof. Since every point in $\mathrm{Prim}(A) = X$ is closed and separated, K_A is an essential ideal in $M_{\rm loc}(A)$, by Proposition 3.1. By Theorem 3.2, $M_{\rm loc}(M_{\rm loc}(A)) = M(K_A)$. To prove the statement of the theorem it thus suffices to find an element in $M(K_A)$ not contained in $M_{\rm loc}(A)$.

Note that every non-empty open subset $O \subseteq X$ contains an open subset which has non-empty boundary. This follows from the above-mentioned fact and the assumption that O is second countable, locally compact Hausdorff and hence metrisable. Therefore, if O were extremally disconnected, it had to be discrete in contradiction to the hypothesis that X is perfect.

Let $\{V_n' \mid n \in \mathbb{N}\}$ be a countable basis for the topology of X. For each $n \in \mathbb{N}$, choose an open subset V_n of X such that $\overline{V_n} \subseteq V_n'$ and $\overline{V_n}$ is not open. Put $W_n = X \setminus \overline{V_n}$. Then $O_n = V_n \cup W_n$ is a dense open subset of X.

Let z_n denote the equivalence class of $\chi_{V_n} \otimes 1 \in C_b(O_n, M(B)_\beta) = M(C_0(O_n) \otimes B)$ in Z. Let $(e_n)_{n\in\mathbb{N}}$ be a strictly increasing approximate identity of B with the properties $e_n e_{n+1} = e_n$ and $\|e_{n+1} - e_n\| = 1$ for all n; see [**16**, Lemma 1.2.3], e.g. Put $p_1 = e_1$, $p_n = e_n - e_{n-1}$ for $n \geq 2$. Then $(p_{2n})_{n\in\mathbb{N}}$ is a sequence of mutually orthogonal positive norm-one elements in B. Set $q_n = \sum_{j=1}^n z_j p_{2j}$, $n \in \mathbb{N}$, where we identify an element $b \in M(B)$ canonically with the constant function in $M(A) = C_b(X, M(B)_\beta)$. By means of this we obtain an increasing sequence $(q_n)_{n\in\mathbb{N}}$ of positive elements in $M_{\rm loc}(A)$ bounded by 1. Since the injective envelope is monotone complete [**13**], the supremum of this sequence exists in $I(A)$ and is a positive element of norm 1, which we will write as $q = \sup_n q_n = \sum_{n=1}^\infty z_n p_{2n}$.

Suppose that $q \in M_{\rm loc}(A)$. Then, for given $0 < \varepsilon < 1/2$, there are a dense open subset $U \subseteq X$ and $m \in C_b(U, M(B)_\beta)_+$ with $\|m\| \leq 1$ such that $\|m - q\| < \varepsilon$. Upon multiplying both on the left and on the right by $p_{2n}^{1/2}$ we find that

$$\sup_{t\in U\cap O_n} \big\|p_{2n}^{1/2} m(t) p_{2n}^{1/2} - \chi_{V_n}(t) p_{2n}^2\big\| = \big\|p_{2n}^{1/2} m p_{2n}^{1/2} - z_n p_{2n}^2\big\| < \varepsilon.$$

Let $n \in \mathbb{N}$ be such that $V_n' \subseteq U$. Define $f_n \in C_b(U)$ by $f_n(t) = \|p_{2n}^{1/2} m(t) p_{2n}^{1/2}\|$, $t \in U$ (note that $p_{2n}^{1/2} m p_{2n}^{1/2} \in C_b(U, B)$). Then $0 \leq f_n \leq 1$ and

$$\begin{aligned} \big|f_n(t) - \chi_{V_n}(t)\big| &= \big|\|p_{2n}^{1/2} m(t) p_{2n}^{1/2}\| - \chi_{V_n}(t)\|p_{2n}^2\|\big| \\ &\leq \big\|p_{2n}^{1/2} m(t) p_{2n}^{1/2} - \chi_{V_n}(t) p_{2n}^2\big\| < \varepsilon \end{aligned}$$

for all $t \in U \cap O_n$. By construction, $\overline{V_n}$ is not open; hence $\partial\overline{V_n} \neq \emptyset$. Each $t_0 \in \partial\overline{V_n}$ also belongs to $\overline{W_n \cap V_n'}$ as $\partial\overline{V_n} = \partial W_n$ and hence $t_0 \in \overline{W_n} \cap V_n' \subseteq \overline{W_n \cap V_n'}$ since V_n' is open. For every $t \in V_n$, $|f_n(t) - 1| < \varepsilon$ and hence $f_n(t) \geq 1 - \varepsilon > 1/2$ for all $t \in \overline{V_n}$, by continuity of f_n. In particular, $f_n(t_0) > 1/2$. For every $t \in W_n \cap V_n'$, we have $f_n(t) < \varepsilon < 1/2$ and thus $f_n(t_0) \leq \varepsilon < 1/2$. This contradiction shows that $q \notin M_{\rm loc}(A)$.

In order to prove that q belongs to $M(K_A)$ it suffices to show that $qa \in K_A$ for every $a \in A$, by Lemma 2.3. For each $n \in \mathbb{N}$ and $a \in A$, $q_n a \in {}^{\rm c}A$ since $z_j p_{2j} a \in ZA$. Therefore, $q_n \in M({}^{\rm c}A)$ for each n. Note that ${}^{\rm c}A$ contains a strictly positive element h. Indeed, taking an increasing approximate identity $(g_n)_{n\in\mathbb{N}}$ of $C_0(X)$ we obtain an increasing approximate identity $u_n = g_n \otimes e_n$, $n \in \mathbb{N}$ of A. It follows easily that $(u_n)_{n\in\mathbb{N}}$ is an approximate identity for ${}^{\rm c}A = \overline{AZ}$. It is well-known that $h = \sum_{n=1}^\infty 2^{-n} u_n$ is then a strictly positive element.

As a result, in order to prove that $(q_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $M({}^{\rm c}A)_\beta$, we only need to show that $(q_n h)_{n\in\mathbb{N}}$ is a Cauchy sequence. For $k \in \mathbb{N}$, $p_{2j} e_k = (e_{2j} - e_{2j-1}) e_k = 0$ if $2j > k + 1$. Consequently,

$$z_j p_{2j} h = \sum_{k=1}^\infty 2^{-k} z_j p_{2j} u_k = \sum_{k=1}^\infty 2^{-k} g_k z_j p_{2j} e_k \qquad (j \in \mathbb{N})$$

yields that, for each $n \in \mathbb{N}$,

$$\begin{aligned} q_n h &= \sum_{j=1}^n \sum_{k=1}^\infty 2^{-k} g_k z_j p_{2j} e_k \\ &= \sum_{k=1}^\infty 2^{-k} g_k z_1 p_2 e_k + \sum_{k=3}^\infty 2^{-k} g_k z_2 p_4 e_k + \ldots + \sum_{k=2n-1}^\infty 2^{-k} g_k z_n p_{2n} e_k. \end{aligned}$$

We conclude that, for $m > n$,

$$\|(q_m - q_n)h\| = \Big\|\sum_{j=n+1}^m \sum_{k=2j-1}^\infty 2^{-k} g_k z_j p_{2j} e_k\Big\| = \max_{n+1\leq j\leq m} \Big\|\sum_{k=2j-1}^\infty 2^{-k} g_k z_j p_{2j} e_k\Big\| \leq \sum_{k=2n+1}^\infty 2^{-k}$$

since $g_k z_j p_{2j} e_k \, g_\ell z_i p_{2i} e_\ell = 0$ for all k, ℓ whenever $i \neq j$; therefore $\|(q_m - q_n)h\| \to 0$ as $n \to \infty$. This proves that $(q_n)_{n\in\mathbb{N}}$ is a strict Cauchy sequence in $M({}^{\rm c}A)$. Let $\tilde{q} \in M({}^{\rm c}A)$ denotes its limit, which is a positive element of norm at most one since $M({}^{c}A)_+$ is closed in the strict topology. In order to show that $\tilde{q} = q$ note at first that $I(M({}^{c}A)) = I({}^{c}A) = I(A)$ by [**4**, Proposition 2.14]. The mutual orthogonality of the p_{2n}'s yields $qq_n = q_m q_n$ for all $m \geq n$. Thus, for all $a \in {}^{c}A$, $aqq_n = aq_m q_n$ which implies that $aqq_n = a\tilde{q}q_n$ for all a. As A is essential in $I(A)$, it follows that $qq_n = \tilde{q}q_n$ for all $n \in \mathbb{N}$ by [**4**, Theorem 3.4]. Repeating the same argument using the strict convergence of $(q_n)_{n\in\mathbb{N}}$ we obtain that $q\tilde{q} = \tilde{q}^2$.

For all $1 \leq n \leq m$, $q_n \leq q_m$ and hence $a^* q_n a \leq a^* q_m a$ for every $a \in {}^{c}A$. It follows that, for all n, $a^* q_n a \leq a^* \tilde{q} a$ for every a and therefore $q_n \leq \tilde{q}$ for all $n \in \mathbb{N}$. Consequently, $q \leq \tilde{q}$. Together with the above identity $(\tilde{q} - q)\tilde{q} = 0$ this entails that $q = \tilde{q} \in M({}^{c}A)$.

Finally, for each $a \in A$, we have $qa \in {}^{c}A \subseteq K_A$. This completes the proof. □

Remark 3.4. A space X as in Theorem 3.3 is perfect if and only if it contains a dense G_δ subset with empty interior. In [**4**, Theorem 6.13], the existence of a dense G_δ subset with empty interior in the primitive spectrum, which was assumed to be Stonean, was used to obtain a C^*-algebra A such that $M_{\rm loc}(A)$ is a proper subalgebra of $I(A)$ and the latter agreed with $M_{\rm loc}(M_{\rm loc}(A))$. In contrast to this example, and also the one considered in [**7**], our approach in Theorem 3.3 does not need any additional information on the injective envelope; nevertheless all higher local multiplier algebras coincide by Theorem 3.2.

Remark 3.5. Taking the two results Corollary 4.8 and Theorem 3.3 together we obtain the following, maybe surprising dichotomy for a compact Hausdorff space X satisfying the assumptions in (3.3). Let $A = C(X) \otimes B$ for a unital, separable, simple C^*-algebra B. Then $M_{\rm loc}(A) = M_{\rm loc}(M_{\rm loc}(A))$. But if we stabilise A to $A_s = A \otimes K(H)$ then $M_{\rm loc}(A_s) \neq M_{\rm loc}(M_{\rm loc}(A_s))$!

With a little more effort we can replace the commutative C^*-algebra in Theorem 3.3 by a nuclear one, provided the properties of the primitive ideal space are preserved. We shall formulate this as a necessary condition on a C^*-algebra A with tensor product structure to enjoy the property $M_{\rm loc}(M_{\rm loc}(A)) = M_{\rm loc}(A)$. Note that, whenever B and C are separable C^*-algebras and at least one of them is nuclear, the primitive ideal space $\mathrm{Prim}(C \otimes B)$ is homeomorphic to $\mathrm{Prim}(C) \times \mathrm{Prim}(B)$, by [**20**, Theorem B.45], for example.

Some elementary observations are collected in the next lemma in order not to obscure the proof of the main result.

Lemma 3.6. *Let X be a topological space, and let $G \subseteq X$ be a dense subset consisting of closed points.*
(i) *If X is perfect then G is perfect (in itself).*
(ii) *For each $V \subseteq X$ open, $\overline{V} \cap G = \overline{V \cap G}^{G}$, where ${}^{-G}$ denotes the closure relative to G.*
(iii) *For each $V \subseteq X$ open, $\partial\big(\overline{V \cap G}^{G}\big) = \partial\overline{V} \cap G$.*

Proof. Assertion (i) is immediate from the density of G and the hypothesis that $X \setminus \{t\}$ is open for each $t \in G$. In (ii), the inclusion "$\supseteq$" is evident. The other inclusion "$\subseteq$" follows from the density of G.

To verify (iii), we conclude from (ii) that

$$G \setminus \overline{V \cap G}^{G} = G \setminus (\overline{V} \cap G) = G \cap (X \setminus \overline{V})$$

and therefore, with $W = X \setminus \overline{V}$,

$$\overline{G \setminus \overline{V \cap G}^G}^G = \overline{G \cap W}^G = G \cap \overline{W},$$

where we used (ii) another time. This entails

$$\partial(\overline{V \cap G}^G) = \overline{V \cap G}^G \cap \overline{G \setminus \overline{V \cap G}^G}^G = \overline{V} \cap G \cap \overline{X \setminus \overline{V}} = \partial\overline{V} \cap G$$

as claimed. □

THEOREM 3.7. *Let B and C be separable C^*-algebras and suppose that at least one of them is nuclear. Suppose further that B is simple and non-unital and that* $\mathrm{Prim}(C)$ *contains a dense G_δ subset consisting of closed points. Let $A = C \otimes B$. If $M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ then* $\mathrm{Prim}(C)$ *contains an isolated point.*

Proof. Let $X = \mathrm{Prim}(C) = \mathrm{Prim}(A)$. We shall assume that X is perfect and conclude from this that $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \neq M_{\mathrm{loc}}(A)$. By Proposition 3.1, K_A is an essential ideal in $M_{\mathrm{loc}}(A)$. Using Theorem 3.2 we are left with the task to find an element in $M(K_A) \setminus M_{\mathrm{loc}}(A)$.

The hypothesis on X combined with the separability assumption yields a dense G_δ subset $S \subseteq X$ consisting of closed separated points which is a Polish space. By Lemma 3.6 (i), S is a perfect metrisable space and therefore cannot be extremally disconnected, as mentioned before. Since a non-empty open subset of a perfect space is clearly perfect, it follows that every non-empty open subset of S contains an open subset which has non-empty boundary.

Let $\{V'_n \mid n \in \mathbb{N}\}$ be a countable basis for the topology of X. For each $n \in \mathbb{N}$, choose an open subset V_n of X such that $\overline{V_n \cap S}^S \subseteq V'_n \cap S$ and $\overline{V_n \cap S}^S$ is not open. By Lemma 3.6 (ii), $\overline{V_n \cap S}^S = \overline{V_n} \cap S$ and we shall use the latter, simpler notation. Put $W_n = X \setminus \overline{V_n}$. Then $O_n = V_n \cup W_n$ is a dense open subset of X.

Using the same notation as in the fourth paragraph of the proof of Theorem 3.3 we define the element $q \in I(A)$ by $q = \sum_{n=1}^{\infty} z_n \otimes p_{2n}$. The argument showing that $q \in M(K_A)$ takes over verbatim from the proof of Theorem 3.3. We will now modify the argument in the fifth paragraph of that proof.

Suppose that $q \in M_{\mathrm{loc}}(A)$. For $0 < \varepsilon < 1/4$, there are a dense open subset $U \subseteq X$ and an element $m \in M(A(U))_+$ with $\|m\| \leq 1$ such that $\|m - q\| < \varepsilon$. Let $n \in \mathbb{N}$ be such that $V'_n \subseteq U$ and choose $t_0 \in \partial\overline{V_n} \cap S \subseteq U \cap S$ using Lemma 3.6 (iii). Since the ideal $C(U)$ of C corresponding to U is not contained in t_0, there is $c \in C(U)_+$ with $\|c\| = 1$ and $\|c + t_0\| = 1$. As the function $t \mapsto \|c + t\|$ is lower semicontinuous, there is an open subset $V \subseteq U$ containing t_0 such that $\|c + t\| > 1 - \varepsilon$ for $t \in V$. Let $a = c^{1/2} \otimes p_{2n}^{1/2} \in C(U) \otimes B = A(U)$ and put $f(t) = \|ama + t\|$, $t \in U$. By [**5**, Lemma 6.4], f is continuous on $U \cap S$ because $ama \in A$. For each $t \in V \cap O_n$ we have

$$\begin{aligned}
\big|f(t) - \chi_{V_n}(t)\big| &\leq \big|\|ama + t\| - \chi_{V_n}(t)\,\|c + t\|\big| + \big|\chi_{V_n}(t)\,\|c + t\| - \chi_{V_n}(t)\big| \\
&\leq \big\|ama + t - \chi_{V_n}(t)c \otimes p_{2n}^2 + t\big\| + (1 - \|c + t\|)\,\chi_{V_n}(t) \\
&\leq \big\|(c^{1/2} \otimes p_{2n}^{1/2})\,m\,(c^{1/2} \otimes p_{2n}^{1/2}) - cz_n \otimes p_{2n}^2\big\| + \varepsilon \\
&\leq \|m - q\| + \varepsilon < 2\,\varepsilon,
\end{aligned}$$

since $(c^{1/2} \otimes p_{2n}^{1/2})\,q\,(c^{1/2} \otimes p_{2n}^{1/2}) = cz_n \otimes p_{2n}^2$. For each $t \in V_n \cap S$ we have $f(t) > 1 - 2\varepsilon > 1/2$ and therefore $f(t_0) > 1/2$ by continuity of f on $U \cap S$ and the fact that $\overline{V_n \cap S}^S = \overline{V_n} \cap S$ by Lemma 3.6 (ii), thus $t_0 \in \overline{V \cap V_n \cap S}^S$. On the other hand,

$$t_0 \in \overline{W_n} \cap V \cap S \subseteq \overline{W_n \cap V} \cap S = \overline{W_n \cap V \cap S}^S$$

as V is open and using Lemma 3.6 (ii) again. Thus $f_n(t_0) \leq 2\varepsilon < 1/2$. This contradiction shows that $q \notin M_{\rm loc}(A)$, and the proof is complete. □

We can now formulate an if-and-only-if condition characterising when the second local multiplier algebra is equal to the first.

Corollary 3.8. *Let $A = C \otimes B$ for two separable C^*-algebras B and C satisfying the conditions of Theorem 3.7. Suppose that* $\mathrm{Prim}(A)$ *contains a dense G_δ subset consisting of closed points. Then $M_{\rm loc}(A) = M_{\rm loc}(M_{\rm loc}(A))$ if and only if* $\mathrm{Prim}(A)$ *contains a dense subset of isolated points.*

Proof. Let $X = \mathrm{Prim}(A)$, X_1 the set of isolated points in X and $X_2 = X \setminus \overline{X_1}$. Then X_1 and X_2 are open subsets of X with corresponding closed ideals $I_1 = A(X_1)$ and $I_2 = A(X_2)$ of A. If X_1 is dense, I_1 is the minimal essential closed ideal of A so $M_{\rm loc}(A) = M(I_1)$. It follows that

$$M_{\rm loc}(M_{\rm loc}(A)) = M_{\rm loc}(M(I_1)) = M_{\rm loc}(I_1) = M_{\rm loc}(A).$$

In the general case, $M_{\rm loc}(A) = M_{\rm loc}(I_1) \oplus M_{\rm loc}(I_2)$ by [**2**, Lemmas 3.3.4 and 3.3.6]. If $X_2 \neq \emptyset$, it contains a dense G_δ subset of closed points and so $I_2 = C(X_2) \otimes B$ satisfies all the assumptions in Theorem 3.7 while X_2 is a perfect space. It follows that

$$\begin{aligned} M_{\rm loc}(M_{\rm loc}(A)) &= M_{\rm loc}(M_{\rm loc}(I_1) \oplus M_{\rm loc}(I_2)) = M_{\rm loc}(M_{\rm loc}(I_1)) \oplus M_{\rm loc}(M_{\rm loc}(I_2)) \\ &\neq M_{\rm loc}(I_1) \oplus M_{\rm loc}(I_2) = M_{\rm loc}(A). \qquad \square \end{aligned}$$

4. *A sheaf-theoretic approach*

In [**5**], we developed the basics of a sheaf theory for general C^*-algebras. This enabled us to establish the following formula for $M_{\rm loc}(A)$ in [**5**, Theorem 7.6]:

$$M_{\rm loc}(A) = \mathrm{alg}\varinjlim{}_{T \in \mathcal{T}}\, \Gamma_b(T, \mathsf{A}),$$

where A is the upper semicontinuous C^*-bundle canonically associated to the multiplier sheaf $\mathfrak{M}_A$ of A [**5**, Theorem 5.6] and $\Gamma_b(T, \mathsf{A})$ is the C^*-algebra of bounded continuous sections of A on T. A like description is available for the injective envelope:

$$I(A) = \mathrm{alg}\varinjlim{}_{T \in \mathcal{T}}\, \Gamma_b(T, \mathsf{I}),$$

where the C^*-bundle I corresponds to the injective envelope sheaf $\mathfrak{I}_A$ of A, see [**5**, Theorem 7.7]. These descriptions are compatible, by [**5**, Corollary 7.8]. Since a continuous section is determined by its restriction to a dense subset, the *-homomorphisms $\Gamma_b(T, \mathsf{B}) \to \Gamma_b(T', \mathsf{B})$, $T' \subseteq T$, $T' \in \mathcal{T}$ are injective for any C^*-bundle B and thus isometric. Consequently, an element $y \in M_{\rm loc}(M_{\rm loc}(A))$ is contained in some C^*-subalgebra $\Gamma_b(T, \mathsf{I})$ and will belong to $M_{\rm loc}(A)$ once we find $T' \subseteq T$, $T' \in \mathcal{T}$ such that $y \in \Gamma_b(T', \mathsf{A})$.

Remark 4.1. Let $a \in \Gamma_b(T, \mathsf{A})$ for a separable C^*-algebra A. By applying Lemma 2.1 to the negative of the upper semicontinuous norm function on A, there is always a smaller dense G_δ subset $S \subseteq \mathrm{Sep}(A) \cap T$ on which the restriction of the function $t \mapsto \|a(t)\|$ is continuous.

On the basis of this, we shall obtain a concise proof of an extension of one of Somerset's main results, [**21**, Theorem 2.7], in this section. This extension is twofold: firstly, we replace the assumption of an identity by the more general hypothesis on A to be quasicentral. Secondly, we establish the result for C^*-subalgebras of $M_{\mathrm{loc}}(A)$ containing A.

The following concept was introduced and initially studied by Delaroche [**9**], [**10**]. A C^*-algebra A is called *quasicentral* if no primitive ideal of A contains the centre $Z(A)$ of A. We recall some basic properties of quasicentral C^*-algebras.

REMARK 4.2. Let A be a quasicentral C^*-algebra.

(i) The mapping $\nu\colon \mathrm{Prim}(A) \to \mathrm{Max}(Z(A))$, $t \mapsto t \cap Z(A)$ is well-defined, surjective and continuous.

(ii) The Dauns–Hofmann isomorphism $Z(M(A)) \to C_b(\mathrm{Prim}(A))$, $z \mapsto f_z$ such that $za + t = f_z(t)(a + t)$ for all $a \in A$, $z \in Z(M(A))$ and $t \in \mathrm{Prim}(A)$ maps $Z(A)$ onto $C_0(\mathrm{Prim}(A))$; see [**20**, A.34] and [**9**, Proposition 1].

(iii) Every approximate identity of $Z(A)$ is an approximate identity for A and thus $A = Z(A)A$; see [**6**, Proposition 1].

Part (i) of the result below on the existence of local identities is already contained in [**9**, Proposition 2] but we provide an independent brief proof along the lines of the proof of [**6**, Theorem 5].

LEMMA 4.3. *Let A be a quasicentral C^*-algebra, $C \subseteq \mathrm{Prim}(A)$ compact and $t \in C$.*

(i) *There exists $z \in Z(A)_+$, $\|z\| = 1$ such that $z + s = 1_{A/s}$, the identity in the primitive quotient A/s for all $s \in C$.*

(ii) *Let U_1 be an open neighbourhood of t contained in C and let $U_2 = \mathrm{Prim}(A) \setminus \overline{U_1}$. If $z \in Z(A)_+$ is as in* (i) *then $z + A(U_2)$ is the identity in $A/A(U_2)$.*

Proof. As $\mathrm{Max}(Z(A))$ is a locally compact Hausdorff space, there is $f \in C_0(\mathrm{Max}(Z(A)))_+$ with $\|f\| = 1$ such that $f(\nu(s)) = 1$ for all $s \in C$ [**19**, 1.7.5]. Identifying $Z(A)$ with $C_0(\mathrm{Prim}(A))$, see Remark 4.2 above, we obtain $z \in Z(A)_+$, $\|z\| = 1$ such that $f_z = f \circ \nu$ and hence $z + s = 1_{A/s}$ for all $s \in C$. This proves (i).

Now let U_1 be an open neighbourhood of t contained in C and put $U_2 = \mathrm{Prim}(A) \setminus \overline{U_1}$. Let $z \in Z(A)_+$ be as in (i). Then $\overline{U_1} = \{s \in \mathrm{Prim}(A) \mid A(U_2) \subseteq s\}$ is homeomorphic to $\mathrm{Prim}(A/A(U_2))$ via $s \mapsto s/A(U_2)$ [**18**, 4.1.11]. Therefore, any identity which holds in $\big(A/A(U_2)\big)/\big(s/A(U_2)\big)$ for a dense set of s holds in $A/A(U_2)$. Since $\big(A/A(U_2)\big)/\big(s/A(U_2)\big) \cong A/s$ and $z + s = 1_{A/s}$ for all $s \in U_1$, it follows that $z + A(U_2) = 1_{A/A(U_2)}$ as claimed in (ii). □

With the help of Lemma 4.3 we can extend a key result in [**5**], viz. [**5**, Lemma 6.9], from the unital case to the situation of quasicentral C^*-algebras. Though the proof is similar, we include the details for completeness.

PROPOSITION 4.4. *Let A be a quasicentral C^*-algebra, and let $t \in \mathrm{Prim}(A)$ be a closed and separated point. Then the natural mapping $\varphi_t\colon \mathsf{A}_t \to A/t$ is an isomorphism.*

Proof. Since A is quasicentral, the C^*-algebra A/t is unital, and since t is a closed point, A/t is simple. Therefore the natural mapping $\varphi_t\colon \mathsf{A}_t \to M_{\mathrm{loc}}(A/t)$ given by [**5**, Proposition 6.2] simplifies to $\varphi_t\colon \mathsf{A}_t \to A/t$. As t is a separated point, $\ker \iota_t = t$ where $\iota_t\colon A \to \mathsf{A}_t$ is the canonical map [**5**, Proposition 6.5]. Since $\varphi_t \circ \iota_t = \pi_t$, where π_t is the canonical surjection $A \to A/t$, we find that φ_t is injective when restricted to $\iota_t(A)$.

Let U be an open neighbourhood of t in $\mathrm{Prim}(A)$, and take $m \in M(A(U))$. Let C be a compact neighbourhood of t contained in U [**18**, 4.4.4]. By Lemma 4.3, there is $z \in Z(A)_+$, $\|z\| = 1$ such that $z + s = 1_{A/s}$ for all $s \in C$. Choose $e \in A(U)_+$ with the property that $\|e\| = 1$ and $e + t = z + t$ in A/t. (Note that $A(U) + t/t = A/t$ as A/t is simple.) Since $N(z - e)(t) = 0$ and $N(z - e)$ is continuous at t, as t is a separated point [**5**, Lemma 6.4], there is an open neighbourhood U_1 of t contained in C such that $N(z - e)(s) < 1/2$ for every $s \in U_1$. Set $Y = \overline{U_1}$ and $U_2 = \mathrm{Prim}(A) \setminus Y$. By Lemma 4.3 (ii), $z + A(U_2)$ is the identity of $A/A(U_2)$. Since $A(U_1)$ sits as an essential ideal in $A/A(U_2)$, we have an embedding of unital C^*-algebras $A/A(U_2) \subseteq M(A(U_1)) = \mathfrak{M}_A(U_1)$. The set $\{s \in \mathrm{Prim}(A) \mid N(z - e)(s) \leq 1/2\}$ is closed in $\mathrm{Prim}(A)$ and contains U_1; consequently $N(z - e)(s) \leq 1/2$ for every $s \in Y$.

Since $N_{A/A(U_2)}\big((z - e) + A(U_2)\big)(s) = N_A(z - e)(s) \leq 1/2$ for every $s \in Y$, we get that $\|1_{A/A(U_2)} - e + A(U_2)\| = \|(z - e) + A(U_2)\| \leq 1/2 < 1$, and thus $e + A(U_2)$ is invertible in $A/A(U_2)$. Take any $y \in A$ such that $y + A(U_2) = (e + A(U_2))^{-1}$. Then we have

$$m_{|\mathfrak{M}_A(U_1)} = m_{|\mathfrak{M}_A(U_1)}(e + A(U_2))(y + A(U_2)) = (me + A(U_2))(y + A(U_2)) \in A/A(U_2),$$

since $me \in A(U) \subseteq A$. As a result, $m_{|\mathfrak{M}_A(U_1)}$ belongs to the image of the map $A \to \mathfrak{M}_A(U_1)$. We thus find that the image of m in $\mathsf{A}_t = \varinjlim_{t \in W} \mathfrak{M}_A(W)$ belongs to the image of the map $A \to \mathsf{A}_t$, and it turns out that the map $A/t \to \mathsf{A}_t$ is surjective. Since it is also injective, we conclude that it is an isomorphism, and so its inverse, φ_t, must be an isomorphism too. □

The following example shows that the statement of Proposition 4.4 can fail if the C^*-algebra is not quasicentral.

Example 4.5. Let $B = C_b(\mathbb{N}, M_2(\mathbb{C}))$ be the C^*-algebra of all bounded (continuous) functions from $\mathbb{N}$ to the 2×2 complex matrices. We shall write elements of B as $x = (x(n))_{n \in \mathbb{N}}$. Let A be the C^*-subalgebra of B consisting of those x such that $x_{ij}(n) \to 0$, $n \to \infty$ for $(i, j) \neq (1, 1)$ and $x_{11}(n) \to \mu(x)$, $n \to \infty$. Then A is a non-unital separable 2-subhomogeneous C^*-algebra with Hausdorff primitive spectrum. In fact, the primitive ideals of A are given by $t_\infty = \ker \mu$ and, for each $n \in \mathbb{N}$, $t_n = \{x \in A \mid x(n) = 0\}$ (with corresponding irreducible representations given by $\pi_\infty \colon A \to \mathbb{C}$, $\pi_\infty(x) = \mu(x)$ and $\pi_n \colon A \to M_2(\mathbb{C})$, $\pi_n(x) = x(n)$, $x \in A$). Clearly $\mathrm{Prim}(A)$ is homeomorphic to the one-point compactification $\mathbb{N}_\infty$ of $\mathbb{N}$, since $\{U_n \mid n \in \mathbb{N}\}$ with $A(U_n) = \bigcap_{j=1}^n t_j$ forms a neighbourhood basis for t_∞.

As $C_0(\mathbb{N}, M_2(\mathbb{C})) = t_\infty$, t_∞ is an essential ideal of A and $M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(t_\infty) = M(t_\infty) = B = I(A)$. Moreover, $M(A)$ consists of those x satisfying $\lim_n x_{12}(n) = \lim_n x_{21}(n) = 0$, $\lim_n x_{11}(n) = \mu(x)$ and $(x_{22}(n))_{n \in \mathbb{N}}$ is bounded. It follows that $\mathrm{Prim}(M(A)) = \beta\mathbb{N} \cup \{t_\infty\}$, where all the ultrafilters in $\beta\mathbb{N}$ yield characters of $M(A)$ via $\lim_{\mathcal{U}} x_{22}(n)$. Any open neighbourhood of t_∞ in $\mathrm{Prim}(M(A))$ must contain one of the U_n's and hence t_m for $m \geq n + 1$. As $\mathbb{N}$ is dense in $\beta\mathbb{N}$ we conclude that no point in $\beta\mathbb{N} \setminus \mathbb{N}$ can be separated from t_∞.

This leads to the following description of the associated upper semicontinuous C^*-bundle. For each $n \in \mathbb{N}$, $\mathsf{A}_{t_n} \cong A/t_n = M_2(\mathbb{C})$. On the other hand, $\mathsf{A}_{t_\infty} = \varinjlim_n M(A(U_n))$ with the connecting mappings given by

$$\big(0, \dots, 0, y(n+1), y(n+2), \dots\big) \longmapsto \big(0, \dots, 0, 0, y(n+2), \dots\big)$$

taking into account that $M(A(U_n)) \cong M(A)$ for each n. It follows that A_{t_∞} is indeed commutative and isomorphic to $C(\{t_\infty\} \cup \beta\mathbb{N} \setminus \mathbb{N}) = \mathbb{C} \times \ell^\infty/c_0$. As a result, the homomorphism $\varphi_{t_\infty} \colon \mathsf{A}_{t_\infty} \to A/t_\infty = \mathbb{C}$ is far from being injective. Note that $t_\infty \supseteq Z(A) \cong c_0$ so that A is not quasicentral.

To complete the picture we note that, in the C^*-bundle I associated to the injective envelope sheaf, the fibres are $\mathsf{I}_{t_n} = M_2(\mathbb{C})$, $n \in \mathbb{N}$ and $\mathsf{I}_{t_\infty} = M_2(\ell^\infty/c_0)$ with the embedding $\mathsf{A}_{t_\infty} \to \mathsf{I}_{t_\infty}$ simply the diagonal map.

A quasicentral C^*-algebra A is said to be *central* if the mapping ν of Remark 4.2 (i) is injective. Since this is equivalent to the hypothesis that A has Hausdorff primitive spectrum [**9**, Proposition 3], the same arguments as in Theorem 6.10 and Corollary 6.11 of [**5**] yield the following consequence.

COROLLARY 4.6. *Let A be a central separable C^*-algebra. Then all the fibres $A_t = A/t$, $t \in \mathrm{Prim}(A)$ are isomorphic to the fibres A_t associated to the multiplier sheaf $\mathfrak{M}_A$ of A. Indeed, the multiplier sheaf $\mathfrak{M}_A$ of A is isomorphic to the sheaf $\Gamma_b(-,\mathsf{A})$ of bounded continuous local sections of the C^*-bundle A associated to $\mathfrak{M}_A$.*

Every C^*-algebra A contains a largest quasicentral ideal J_A, which is the intersection of all closed ideals in A that contain $Z(A)$ [**10**, Proposition 1]. Clearly, the hypothesis in our main result of this section below is equivalent to the assumption that J_A is essential.

THEOREM 4.7. *Let A be a separable C^*-algebra such that $\mathrm{Prim}(A)$ contains a dense G_δ subset consisting of closed points. Suppose A contains a quasicentral essential closed ideal. If B is a C^*-subalgebra of $M_{\mathrm{loc}}(A)$ containing A then $M_{\mathrm{loc}}(B) \subseteq M_{\mathrm{loc}}(A)$. In particular, $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$.*

Proof. As $M_{\mathrm{loc}}(I) = M_{\mathrm{loc}}(A)$ for every $I \in \mathscr{I}_{\mathrm{ce}}(A)$, we can assume without loss of generality that A itself is quasicentral.

Take $y \in M(J)$ for some $J \in \mathscr{I}_{\mathrm{ce}}(B)$, and let $T \in \mathcal{T}$ be such that $y \in \Gamma_b(T, \mathsf{I})$ (recall that $M_{\mathrm{loc}}(B) \subseteq I(B) = I(A)$). By hypothesis, and the fact that $\mathrm{Sep}(A)$ itself is a dense G_δ subset, we can assume that T consists of closed separated points of $\mathrm{Prim}(A)$. Take $h \in J$ with the property that $hz \neq 0$ for every non-zero projection $z \in Z$ (Lemma 2.4). By Remark 4.1, there is $S \in \mathcal{T}$ contained in T such that the function $t \mapsto \|h(t)\|$ is continuous when restricted to S (viewing h as a section in $\Gamma_b(S, \mathsf{A})$). Consequently, the set $S' = \{t \in S \mid h(t) \neq 0\}$ is open in S and intersects every $U \in \mathcal{D}$ non-trivially; it is thus a dense G_δ subset of $\mathrm{Prim}(A)$. Replacing T by S' if necessary, we may assume that $h(t) \neq 0$ for all $t \in T$.

A standard argument yields a separable C^*-subalgebra B' of J containing AhA and such that $yB' \subseteq B'$ and $B'y \subseteq B'$. Let $\{b_n \mid n \in \mathbb{N}\}$ be a countable dense subset of B'. For each n, let $T_n \in \mathcal{T}$ be such that $b_n \in \Gamma_b(T_n, \mathsf{A})$. Letting $T' = \bigcap_n T_n \cap T \in \mathcal{T}$ we find that $B' \subseteq \Gamma_b(T', \mathsf{A})$ and hence $B'_t = \{b(t) \mid b \in B'\} \subseteq \mathsf{A}_t$ for each $t \in T'$.

For each $t \in T$, the C^*-algebras A_t and A/t are isomorphic, by Proposition 4.4 above, and since A/t is unital and simple (as t is closed), we obtain $\mathsf{A}_t\, h(t)\, \mathsf{A}_t = \mathsf{A}_t$ for each $t \in T'$. Consequently,

$$\mathsf{A}_t = \mathsf{A}_t\, h(t)\, \mathsf{A}_t = (A/t)h(t)(A/t) = A_t h(t) A_t = (AhA)_t \subseteq B'_t$$

and thus $B'_t = \mathsf{A}_t$ for all $t \in T'$. We can therefore find, for each $t \in T'$, an element $b_t \in B'$ such that $b_t(t) = 1(t)$. It follows that $y(t) = y(t)\,1(t) = (yb_t)(t) \in \mathsf{A}_t$ for all $t \in T'$, which yields $y \in \Gamma_b(T', \mathsf{A})$. This proves that $y \in M_{\mathrm{loc}}(A)$. □

COROLLARY 4.8. *For every central separable C^*-algebra A, $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$.*

In [**17**], Pedersen showed that every derivation of a separable C^*-algebra A becomes inner in $M_{\mathrm{loc}}(A)$ when extended to the local multiplier algebra. His question whether every derivation of $M_{\mathrm{loc}}(A)$ is inner (when A is separable) has since been open and seems to be connected to the problem how much bigger $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ can be. In this direction, Somerset proved the next result in [**21**, Theorem 2.7] in the unital case. Our approach shows that it is an immediate consequence of Pedersen's theorem, in view of Theorem 4.7 above.

COROLLARY 4.9. *Let A be a quasicentral separable C^*-algebra such that* $\mathrm{Prim}(A)$ *contains a dense G_δ subset consisting of closed points. Then every derivation of $M_{\mathrm{loc}}(A)$ is inner.*

Proof. Let $d\colon M_{\mathrm{loc}}(A) \to M_{\mathrm{loc}}(A)$ be a derivation. Let B be a separable C^*-subalgebra of $M_{\mathrm{loc}}(A)$ containing A which is invariant under d. By [**2**, Theorem 4.1.11], $d_B = d_{|B}$ can be uniquely extended to a derivation $d_{M_{\mathrm{loc}}(B)}\colon M_{\mathrm{loc}}(B) \to M_{\mathrm{loc}}(B)$. Both derivations can be uniquely extended to their respective injective envelopes, by [**14**, Theorem 2.1], but since $I(B) = I(M_{\mathrm{loc}}(B))$, we have $d_{I(B)} = d_{I(M_{\mathrm{loc}}(B))}$. The same argument applies to the extension of d, since $I(B) = I(A) = I(M_{\mathrm{loc}}(A))$; in other words, $d_{I(M_{\mathrm{loc}}(A))} = d_{I(B)}$ which we will abbreviate to $\tilde{d}$. By [**17**, Proposition 2], $d_{M_{\mathrm{loc}}(B)} = \mathrm{ad}\, y$ for some $y \in M_{\mathrm{loc}}(B)$; in fact, $y \in M_{\mathrm{loc}}(A)$ by Theorem 4.7. By uniqueness, $\tilde{d} = \mathrm{ad}\, y$ and hence $d = \mathrm{ad}\, y$ on $M_{\mathrm{loc}}(A)$. □

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