Sheaves of \( C^* \)-algebras

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Dedicated to the memory of Erhard Schmidt

We develop the basics of a theory of sheaves of \( C^* \)-algebras and, in particular, compare it to the existing theory of \( C^* \)-bundles. The details of two fundamental examples, the local multiplier sheaf and the injective envelope sheaf, are discussed.

\(^*\) Dedicated to the memory of Erhard Schmidt

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1 Introduction

A commutative \( C^* \)-algebra can be described as the algebra of bounded continuous sections, vanishing at infinity, of a bundle of one-dimensional \( C^* \)-algebras over its structure space. There are numerous extensions of this approach in the non-commutative setting, see, e.g., [8], [20], [4]. This theory of \( C^* \)-bundles or fields of \( C^* \)-algebras works best over a (locally compact) Hausdorff base space. In fact, in the non-Hausdorff situation, points become less significant and instead of fibres one should rather think of stalks of \( C^* \)-algebraic sheaves. There appears to be no systematic account on sheaves of \( C^* \)-algebras; thus, part of the objective of this paper is to provide the basic theory in a fairly concise manner.

The open subsets of the primitive spectrum \( \text{Prim}(A) \) of a \( C^* \)-algebra \( A \) correspond to closed ideals of \( A \). However, instead of associating the ideal \( A(U) \) to an open subset \( U \subseteq \text{Prim}(A) \) we propose rather to use the multiplier algebra \( M(A(U)) \). This is certainly justified in the separable case, since, by a result of Larry Brown [6], every isomorphism between the multiplier algebras of two separable \( C^* \)-algebras restricts to an isomorphism between the \( C^* \)-algebras themselves. In the commutative case we therefore associate to every open subset \( U \subseteq X \), when \( A = C_0(X) \), the algebra \( C_0(U) \) of all bounded continuous complex-valued functions on \( U \) and, if \( V \subseteq U \) is another open subset, we have the restriction maps \( C_0(U) \rightarrow C_0(V) \). The stalks of this well-known sheaf of continuous functions on \( X \) are given by \( A_t = \lim \overline{C_0(U)} \), where \( U \) ranges over all open neighbourhoods of \( t \in X \). Denoting by \( [f] \) the equivalence class of \( f \in C_0(U) \) in the \( C^* \)-direct limit, we find that \( [f] \mapsto f(t) \) is an isomorphism and hence \( A_t \cong \mathbb{C} \). In contrast to the algebraic situation, we do not get any “germs” in this case: if \( f(s) = g(s) \) for two continuous functions \( f \) and \( g \) defined on a neighbourhood of \( t \), then \( \| [f] - [g] \| < \varepsilon \) for every \( \varepsilon > 0 \) and so \( [f] = [g] \) in \( A_t \). In this way we recover the usual bundle of \( C^* \)-algebras.

The extension to the non-commutative setting also works with the canonical restriction mappings \( M(A(U)) \rightarrow M(A(V)) \), where \( V \subseteq U \subseteq \text{Prim}(A) \) are open subsets. The stalk at a primitive ideal \( t \in \text{Prim}(A) \) is given by \( A_t = \lim M(I) \), where \( I \) ranges over all closed ideals of \( A \) not contained in \( t \). For such \( I \), we have

\[
M(I) \longrightarrow M(I/t \cap I) \xrightarrow{\cong} M(I + t/t) \longrightarrow M_{\text{loc}}(A/t);
\]
here, $M_{\text{loc}}(B)$ denotes the local multiplier algebra of a $C^*$-algebra $B$, see [1]. Since the above maps are compatible with the restriction homomorphisms, we obtain a mapping

$$\varphi_t: A_t \longrightarrow M_{\text{loc}}(A/t)$$

which will be an isomorphism under favourable circumstances; see Corollary 6.7. The appearance of the local multiplier algebra in this setting is no accident; in fact, part of this study is motivated by trying to understand this construction in more depth using sheaf-theoretic methods, see below.

For any $C^*$-algebra $A$, $M_{\text{loc}}(A)$ can be realised as a continuous $C^*$-bundle over $\text{Glimm}(M_{\text{loc}}(A))$, the Glimm ideal space of $M_{\text{loc}}(A)$, with all fibres being prime $C^*$-algebras [1, Corollary 3.5.11]. If $A$ is separable, then all fibres are even primitive [25, Theorem 3.5]. However, this description does not seem to reveal much of the structure of the local multiplier algebra. For instance, if $A$ is commutative, say $A = C[0, 1]$, then $M_{\text{loc}}(A)$ is a commutative AW*-algebra and hence $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$ [1, Theorem 2.3.8]. In contrast to this, if $A = C[0, 1] \otimes K$, where $K$ denotes the compact operators on separable Hilbert space, then $M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A)$; see [3], [5]. But if $A$ is a unital separable $C^*$-algebra with Hausdorff primitive ideal space then, again, $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$ [25, Theorem 2.7]. A different type of $C^*$-algebras $A$ such that $M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A)$ is exhibited in [2]. It is hoped that a sheaf-theoretic description of $M_{\text{loc}}(A)$, see Section 7, will shed some light on this strange behaviour.

Of course, sheaves of $C^*$-algebras (and Banach algebras and Banach spaces) have been looked at before, notably by Dauns, Hofmann and co-authors ([14], [15], [16], e.g.). These studies, however, tend to take the viewpoint of representation of a $C^*$-algebra by continuous sections of a bundle. Indeed, in [15], an equivalence between certain (pre-)sheaves and bundles is established for completely regular base spaces. Our main new examples—the multiplier sheaf and the injective envelope sheaf, see Section 3—are not generated from an (upper semicontinuous) $C^*$-bundle and thus work better over general, possibly non-Hausdorff base spaces, in particular the primitive ideal space of an arbitrary $C^*$-algebra. We shall discuss the relations with the more traditional bundle approach in Section 6. Another difference of the present work to the existing literature is that we are not striving for a framework that covers very general topological-algebraic structures but focus on the situation of $C^*$-algebras, with the advantage of having more structure available.

After we compile the necessary notation in Section 2, the concepts of a presheaf and a sheaf of $C^*$-algebras are introduced in Section 3. Section 4 contains a brief recollection of the necessary background on $C_0(X)$-algebras and $C^*$-algebras over a topological space. We discuss a concept of an upper semicontinuous $C^*$-bundle over a not necessarily Hausdorff base space in Section 5. Starting from such a bundle $(A, \pi, X)$ one obtains naturally a sheaf of $C^*$-algebras by taking the $C^*$-algebras $\Gamma_{\delta}(U, A)$ of bounded continuous sections on open subsets $U \subseteq X$ (Theorem 5.3). More interestingly, in Theorem 5.6, we associate to every presheaf of $C^*$-algebras over $X$ a canonical upper semicontinuous $C^*$-bundle using the stalks at each $t \in X$. These procedures are not always inverses of each other but in some situations they are. This is in particular the case for sheaves of Banach modules over a sheaf of commutative $C^*$-algebras (Definition 5.8) under various additional hypotheses on the base space $X$ (Propositions 5.10 and 5.12 and Corollary 6.11).

Starting with the multiplier sheaf of a $C^*$-algebra $A$ over a topological space $X$ we obtain in Section 6 certain $C^*$-algebras $A_t$, $t \in X$, which reduce to $M_{\text{loc}}(A/t)$, in the case $X = \text{Prim}(A)$. We investigate the relations between these and the fibres $A_t$ of the bundle introduced above and characterise when $A_t$ and $M_{\text{loc}}(A/t)$ are isomorphic in the case of a separable $C^*$-algebra $A$ (Corollary 6.7). Separated points in the primitive ideal space play an important role in this discussion.

For a commutative $C^*$-algebra $A$, both $M_{\text{loc}}(A)$ and the injective envelope $I(A)$ coincide with the direct limit $\text{alg lim}_{T \in \mathcal{T}} C_b(T)$, where $\mathcal{T}$ denotes the downwards directed set of dense $G_d$ subsets of $\text{Prim}(A)$. Even slight extensions into the non-commutative world—such as tensoring with $K$—destroy such a description; compare [3, Section 6]. By using a derived sheaf $\mathcal{D}(A, \pi, X)$ built from the canonically associated $C^*$-bundle $(A, \pi, X)$ of a sheaf discussed before, we obtain, in Theorem 7.6, the analogous representation $M_{\text{loc}}(A) = \text{alg lim}_{T \in \mathcal{T}} \Gamma_b(T, A)$ for an arbitrary $C^*$-algebra $A$. A strong rigidity property of the injective envelope sheaf is established in Theorem 7.7 stating that it coincides with its derived sheaf, yielding the like formula $I(A) = \text{alg lim}_{T \in \mathcal{T}} \Gamma_b(T, 1)$. 

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2 Notation

For the reader’s benefit, we compile here a list of our notation which we shall use as consistently as possible in the following.

By $A, B, \ldots$ we denote (not necessarily unital) $C^*$-algebras and by $I, J, \ldots$ (two-sided) ideals of them. Since we may have an opportunity to talk about non-closed ideals, we shall always specify explicitly when an ideal (or any other subspace) is closed. The multiplier algebra of a $C^*$-algebra $A$ is designated by $M(A)$, and $ZM(A)$ is its centre. The local multiplier algebra of $A$, $M_{loc}(A)$, is defined as $M_{loc}(A) = \lim_{\leftarrow} M(I)$, where $I$ runs through the (downwards directed) family of closed essential ideals of $A$. For more details on $M_{loc}(A)$, we refer to [1].

Suppose $\{A_\alpha\}$ is a directed system of $C^*$-algebras (suppressing the connecting *-homomorphisms); then $\lim_{\rightarrow} A_\alpha$ denotes their $C^*$-direct limit. The notation $\underline{\lim} A_\alpha$ is reserved for the uncompleted direct limit, that is, the direct limit in the category of complex *-algebras (which, on occasion, will turn out to be already complete). We point out that there is always a canonical *-homomorphism $\underline{\lim} A_\alpha \to \lim_{\rightarrow} A_\alpha$ but, if the connecting maps are not isometries, this may not be injective. The symbol $\prod_{\alpha} A_\alpha$, where $\{A_\alpha\}$ is a family of $C^*$-algebras, stands, of course, for all bounded families $(x_\alpha), x_\alpha \in A_\alpha$, and hence is a $C^*$-algebra.

Topological spaces are typically named $X, Y, \ldots$ and $U, V$ etc. are open subsets of them. For a topological space $X$, the lattice of all open subsets of $X$ is considered as a category in the usual way (that is, there is an arrow from $V$ to $U$ if and only if $V \subseteq U$) and this category is denoted by $O_X$. In general, we shall use calligraphic fonts for categories such as $C^*$ and $C_1^*$ for the category of $C^*$-algebras with *-homomorphisms as the morphisms and for the subcategory of unital $C^*$-algebras with unital *-homomorphisms as the morphisms, respectively. Gothic letters such as $\mathcal{A}, \mathcal{B}$ or $\mathcal{D}$ will be reserved for sheaves and sans serif letters like $A, B$ will denotes bundles.

For a $C^*$-algebra $A$, $\text{Prim}(A)$ denotes the primitive ideal space of $A$ (equipped with the hull-kernel or Jacobson topology), and elements of $\text{Prim}(A)$ will typically be $t, t', \text{ etc.}$ (since we think of these primitive ideals as points in $\text{Prim}(A)$). For elements in $C^*$-algebras, we shall use $a, b$ and sometimes $x, y$ although the latter are more frequently used for points in a topological space. Sections of a bundle (of $C^*$-algebras) will be denoted by $s$, and maps between topological spaces typically go under the names $\varphi$ or $\psi$.

3 Sheaves of $C^*$-algebras

In this section we introduce the concept of a sheaf of $C^*$-algebras and discuss some first examples.

**Definition 3.1** Let $X$ be a topological space. A presheaf of $C^*$-algebras is a contravariant functor $\mathcal{A}: O_X \to C^*$. A sheaf of $C^*$-algebras is a presheaf $\mathcal{A}$ such that $\mathcal{A}(\emptyset) = 0$ and, for every open subset $U$ of $X$ and every open cover $U = \bigcup U_i$, the maps $\mathcal{A}(U) \to \mathcal{A}(U_i)$ are the limit of the diagrams $\mathcal{A}(U_i) \to \mathcal{A}(U_i \cap U_j)$ for all $i, j$.

The “limit of the diagrams” is to be understood in the categorical sense; see, e.g., [17, pp. 68–72]. We shall call the $C^*$-algebra $\mathcal{A}(U)$ the section algebra over $U \in O_X$ and shall denote by $s_{|V}, V \subseteq U$ open, the “restriction” of $s \in \mathcal{A}(U)$ to $V$, that is, the image of $s$ in $\mathcal{A}(V)$ under the *-homomorphism $\mathcal{A}(U) \to \mathcal{A}(V)$. With this notation, the unique gluing property of a sheaf can be expressed as follows: for each bounded compatible family of sections $s_i \in \mathcal{A}(U_i)$, i.e., $s_{|U_i \cap U_j} = s_{|U_i \cap U_j}$ for all $i, j$, there is a unique section $s \in \mathcal{A}(U)$ such that $s_{|U_i} = s_i$ for all $i$. Note also that, in order to be compatible when working with sheaves in $C_1^*$, we understand here that $0 = \mathcal{A}(\emptyset)$ is a unital $C^*$-algebra.

The following is our primary example.

**Example 3.2** Let $A$ be a $C^*$-algebra. By the multiplier sheaf of $A$ we understand the functor $M_A: O_{\text{Prim}(A)} \to C_1^*$ given by $M_A(U) = M(A(U))$, where $M(A(U))$ denotes the multiplier algebra of the closed ideal $A(U)$ of $A$ associated to the open subset $U \subseteq \text{Prim}(A)$. The maps $M(A(U)) \to M(A(V))$, for $V \subseteq U$, are the restriction homomorphisms.

To show that this indeed defines a sheaf, we need a simple lemma.

**Lemma 3.3** Let $I_1$ and $I_2$ be closed ideals in a $C^*$-algebra $A$. Suppose that $x_j \in M(I_j)$ satisfy $x_{1|I_1 \cap I_2} = x_{2|I_1 \cap I_2}$, $j = 1, 2$. Then, for $a \in I_1$ and $b \in I_2$ we have $(ax_1)b = a(x_2b)$.

**Proof.** Let $(e_\alpha)$ be an approximate identity for $I_2$. We have

$$(ax_1)b = \lim_\alpha (e_\alpha(a)x_1)b = \lim_\alpha ((e_\alpha a)x_1)b = \lim_\alpha ((e_\alpha a)x_2)b = \lim_\alpha e_\alpha(a(x_2b)) = a(x_2b).$$

\[ \square \]
**Proposition 3.4** The above functor \( \mathcal{M} \) defines a sheaf of \( C^* \)-algebras.

**Proof.** Since \( \mathcal{M} \) is clearly a presheaf, we need to check the coherence condition. Let \( U = \bigcup U_i \) be an open cover of the open subset \( U \subseteq \text{Prim}(A) \). Denote by \( \rho_{ij} : M(A(U_i)) \to M(A(U_i \cap U_j)) \) and \( \rho_i : M(A(U)) \to M(A(U_i)) \) the corresponding restriction maps. Set

\[
B = \left\{ (x_i) \in \prod_i M(A(U_i)) \mid \rho_{ij}(x_i) = \rho_{ij}(x_j) \text{ for all } i, j \right\}.
\]

Then \( B \) is the limit of the maps \( \{\rho_{ij}\} \) in the category \( C^*_1 \), and we have a \( * \)-homomorphism \( \rho : M(A(U)) \to B \) defined by \( \rho(x) = (\rho_i(x)) \) for every \( x \in M(A(U)) \). We need to show that \( \rho \) is an isomorphism.

Assume that \( \rho_i(x) = 0 \) for every \( i \). Observe that \( A(U) = \bigcup_i A(U_i) \). Since the restriction of \( x \) to \( A(U_i) \) is 0, it follows that \( xa_i = 0 \) for every \( a_i \in A(U_i) \). Therefore \( xa = 0 \) for all \( a \in A(U) \) and thus \( x = 0 \).

In order to verify that \( \rho \) is surjective, let \( (x_i) \in B \). We show by induction on \( n \in \mathbb{N} \) that, for all indices \( i_1, \ldots, i_n \), there is a unique \( x_{i_1, \ldots, i_n} \in M(\bigcap_{k=1}^n A(U_{i_k})) \) such that \( x_{i_1, \ldots, i_n} = x_{i_k}a_{i_k} \) and \( a_{i_k}x_{i_1, \ldots, i_{k-1}} = a_{i_k}x_{i_{k-1}, \ldots, i_n} \) for each \( a_{i_k} \in A(U_{i_k}) \) and all \( k = 1, \ldots, n \). There is nothing to prove for \( n = 1 \). Assume that \( n > 1 \) and that the claim holds for families with less than \( n \) elements. Given \( i_1, \ldots, i_n \) we put \( y_1 = x_{i_1, \ldots, i_{n-1}} \), \( y_2 = x_{i_1, \ldots, i_n} \), \( J_1 = A(U_{i_1}) + \ldots + A(U_{i_{n-1}}) \) and \( J_2 = A(U_{i_n}) \). We are going to show that \( y_1a = y_2a \) for arbitrary \( a \in J_1 \cap J_2 \). Since the lattice of ideals of a \( C^* \)-algebra is distributive, we get

\[
J_1 \cap J_2 = (A(U_{i_1}) \cap A(U_{i_n})) + \ldots + (A(U_{i_{n-1}} \cap A(U_{i_n})),
\]

so that we can write \( a = a_1 + \ldots + a_{n-1} \) with \( a_k \in A(U_{i_k}) \cap A(U_{i_n}) \) for \( k = 1, \ldots, n-1 \). Thus

\[
y_1a = y_1a_1 + \ldots + y_1a_{n-1} = x_{i_1}a_1 + \ldots + x_{i_n}a_{n-1} = x_{i_1}a_1 + \ldots + x_{i_n}a_{n-1} = x_{i_1}a = y_2a,
\]

by induction hypothesis. Similarly, \( ay_1 = ay_2 \) for each \( a \in J_1 \cap J_2 \). We can therefore define a multiplier \( x_{i_1, \ldots, i_n} \) on \( J_1 + J_2 \) by the rule \( x_{i_1, \ldots, i_n}(b_1 + b_2) = y_1b_1 + y_2b_2 \), for \( b_1 \in J_1 \) and \( b_2 \in J_2 \), and similarly for right multiplication. By Lemma 3.3, this gives a well-defined multiplier:

\[
((a_1 + a_2)x_{i_1, \ldots, i_n})(b_1 + b_2) = (a_1y_1 + a_2y_2)(b_1 + b_2) = a_1(y_1b_1 + a_1(y_2b_2) + a_2(y_1b_1) + a_2(y_2b_2) = (a_1 + a_2)(x_{i_1, \ldots, i_n}(b_1 + b_2))
\]

for all \( a_1, b_1 \in J_1 \) and \( a_2, b_2 \in J_2 \).

So far we have shown that, for every finite set \( F \) of indices, the map

\[
M\left(\sum_{i \in F} A(U_i)\right) \to B_F = \{(x_i)_{i \in F} \mid \rho_{ij}(x_i) = \rho_{ij}(x_j) \text{ for all } i, j \in F\}
\]

is a \( * \)-isomorphism. If \( (x_i) \in B \), we therefore get a well-defined multiplier \( x \) on the algebraic sum \( \sum_{i \in F} A(U_i) \). It remains to show that it is a bounded operator. If \( x = \sum_{i \in F} A(U_i) \), for a finite set \( F \) of indices, then, from the \( * \)-isomorphism above, we obtain \( \|x_F\| = \sup_{i \in F} \|x_i\| \leq \|\sum_{i \in F} x_i\| \), so that left (resp., right) multiplication by \( x \) gives a bounded operator on \( \sum_{i \in F} A(U_i) \). It follows that we can extend it to an element in \( M(A(U)) \), as desired.

This concludes the proof. \( \square \)

**Remark 3.5** The algebra \( B \) appearing in the proof of Proposition 3.4 should not be confused with the inverse limit of \( C^* \)-algebras, or pro-\( C^* \)-algebra, discussed in [24]; the latter is not a \( C^* \)-algebra in general. The basic idea above is that, for a pair of closed ideals \( I \) and \( J \) in a \( C^* \)-algebra, \( M(I + J) \) is given by the pullback \( M(I) \otimes_{M(I+J)} M(J) \); compare also diagram (5.3) below. Suppose that \( \{I_\lambda \mid \lambda \in \Lambda\} \) is a family of closed ideals in a \( C^* \)-algebra \( A \) and the closed ideal \( K \) is given by \( K = \sum_\lambda I_\lambda \). For each finite set \( F \subseteq \Lambda \), we put

\[
B_F = \left\{ (x_\lambda) \in \prod_{\lambda \in F} M(I_\lambda) \mid (x_\lambda, x_\mu) \in M(I_\lambda + I_\mu) \text{ for all } \lambda, \mu \in F \right\}.
\]
see [23], in particular 3.11. Then \( M(K) = \lim_F B_F \), where \( F \) runs through the upwards directed set of finite subsets of \( \Lambda \).

Our second example is the injective envelope sheaf of a \( C^* \)-algebra.

**Example 3.6** Let \( A \) be a \( C^* \)-algebra, and let \( I(A) \) denote the injective envelope of \( A \); see [3] or [11]. Recall that for every closed ideal \( I \) of \( A \) there is a unique central open projection \( p_I \) in \( I(A) \) such that \( p_I I(I) \) is the injective envelope of \( I \) [13, Lemma 1.1]. ('Open’ in this context means relative to the regular monotone completion of \( A \)). Define a presheaf \( \mathcal{J}_A \) over \( \text{Prim}(A) \) by assigning to each open subset \( U \) of \( \text{Prim}(A) \) the injective envelope \( p_U I(I) = I(A(U)) \) of \( A(U) \), where \( p_U = p_{A(U)} \). If \( V \subseteq U \), then \( p_V \leq p_U \); hence there is a surjective \( * \)-homomorphism \( I(A(U)) \to I(A(V)) \) given by multiplication by \( p_V \). Note that \( p_U = p_V \) whenever \( U = V \). The set \( \{ p_U \mid U \in \mathcal{O}_{\text{Prim}(A)} \} \) is a complete Boolean algebra isomorphic to the Boolean algebra of regular open subsets of \( \text{Prim}(A) \) [13, Theorem 1.5], and it is precisely the set of projections of the AW*-algebra \( Z(I(A)) \), the centre of \( I(A) \). We will show that \( \mathcal{J}_A \) gives a sheaf of \( C^* \)-algebras.

Suppose that \( U = \bigcup_i U_i \) is a covering of \( U \) by open sets. Then \( p_U = \bigvee_i p_{U_i} \). Let \((x_i)\) be a bounded compatible family, with \( x_i \in p_{U_i} I(A)_+ \) for each \( i \). For each finite set \( F \) of indices, we can find, by finite Boolean algebra, an \( x_F \in \bigvee_{i \in F} p_{U_i} I(A) \) such that \( x_F p_{U_i} \equiv x_i \) for each \( i \in F \). Put \( x = I(A)_{\text{sa-sup}} x_F \). Then \( x \in \bigvee_{i \in F} I(A) \) and \( p(x) = x_F \), since for \( F \subseteq F^c \), both finite, we have \( p_F x_F = x_F \). It follows that \( \mathcal{J}_A \) satisfies the unique gluing axiom.

We conclude this section with a few general remarks on sheaves of \( C^* \)-algebras. The category \( \mathcal{P} \mathcal{S} \mathcal{h}(X) \) of presheaves of \( C^* \)-algebras over \( X \) is the category having as objects the presheaves of \( C^* \)-algebras over \( X \) and as morphisms the natural transformations of contravariant functors \( \mathfrak{A} : \mathcal{O}_X \to \mathcal{C}^* \). The category \( \mathcal{S} \mathcal{h}(X) \) is the full subcategory of \( \mathcal{P} \mathcal{S} \mathcal{h}(X) \) whose objects are the sheaves over \( X \). In case we need to specify the values of a sheaf, we shall write \( \mathcal{S} \mathcal{h}(X, \mathcal{C}) \) where \( \mathcal{C} = C^* \) or \( \mathcal{C} = C^*_t \).

**Remarks 3.7** 1. Let \( f : Y \to X \) be a continuous mapping between the topological spaces \( Y \) and \( X \). Then \( f^* : \mathcal{O}_X \to \mathcal{O}_Y \) given by \( f^*(U) = f^{-1} (U) \) preserves arbitrary infima and suprema, and thus we get a functor \( f_* : \mathcal{P} \mathcal{S} \mathcal{h}(Y) \to \mathcal{P} \mathcal{S} \mathcal{h}(X) \) which sends sheaves over \( Y \) to sheaves over \( X \), the direct image functor.

2. For an open subset \( U \) of \( X \), we can restrict a sheaf over \( X \) to a sheaf over \( U \) and thus get a functor \( \mathcal{S} \mathcal{h}(X) \to \mathcal{S} \mathcal{h}(U) \). This is a simple, special case of the inverse image functor.

3. For a topological space \( X \), a \( C^* \)-algebra over \( X \) is defined to be a pair \((A, \psi)\) consisting of a \( C^* \)-algebra \( A \) and a continuous mapping \( \psi : \text{Prim}(A) \to X \); see, e.g., [18, Definition 2.3]. In such a situation, we obtain a canonical sheaf of \( C^* \)-algebras over \( X \) by taking \( \psi_* (\mathfrak{M}_A) \), where \( \mathfrak{M}_A \) is the multiplier sheaf of \( A \).

4 The Hausdorff case

Sheaves of \( C^* \)-algebras over a Hausdorff base space are expected to be particularly well behaved; this will be discussed in the subsequent Sections 5 and 6. The present section is of a preparatory nature.

Let \( X \) be a locally compact Hausdorff space. By a \( C_0(X) \)-algebra we understand a \( C^* \)-algebra \( A \) together with an essential \( * \)-homomorphism \( \iota : C_0(X) \to ZM(A) \) into the centre \( ZM(A) \) of the multiplier algebra of \( A \); see, e.g., [18] or [20]. (Recall that \( \text{essentiality} \) here means \( \iota(C_0(X))A = A \) or equivalently, by the Cohen–Hewitt factorization theorem [8, Theorem V.9.2], \( \iota(C_0(X)) A = A \).

We make contact between this concept and Remark 3.7.3 via the following well-known fact, see, e.g., [20, [26]. We provide a quick, independent proof suitable for our purposes.

**Proposition 4.1** Let \( X \) be a locally compact Hausdorff space, and let \( A \) be a \( C^* \)-algebra. Then \( A \) is a \( C_0(X) \)-algebra if and only if \((A, \psi)\) is a \( C^* \)-algebra over \( X \) for some continuous mapping \( \psi : \text{Prim}(A) \to X \).

**Proof.** Suppose we are given an essential \( * \)-homomorphism \( \iota : C_0(X) \to ZM(A) \). For \( t \in \text{Prim}(A) \), choose an irreducible representation \( \pi_t : A \to B(H_t) \) on a Hilbert space \( H_t \) such that \( \ker \pi_t = t \) and let \( \pi_t : M(A) \to B(H_t) \) be its unique extension to an irreducible representation of \( M(A) \). The induced \( * \)-homomorphism \( \gamma_t := \pi_t \circ \iota : C_0(X) \to A \) is non-zero since \( \iota(C_0(X))A = A \). Hence, there is a unique \( x \in X \) such that \( \gamma_t(f) = f(x) \) for all \( f \in C_0(X) \) and we can define \( \psi : \text{Prim}(A) \to X \) by \( \psi(t) = x \). In order to show that \( \psi \) is continuous, take
an open subset \( U \) of \( X \). Let

\[
I = \bigcap_{s \notin \psi^{-1}(U)} s,
\]

which is a closed ideal of \( A \). We aim to show that \( \psi^{-1}(U) = U(I) \) and hence \( \psi^{-1}(U) \) is open. If \( s \notin \psi^{-1}(U) \) then \( I \subseteq s \) and therefore \( s \notin U(I) \). If \( \psi(s) \in U \) there is \( f \in C_0(U) \) such that \( f(\psi(s)) \neq 0 \). Since \( \pi_s(\iota(f) a) = f(\psi(s)) \pi_s(a) \) for every \( a \in A \), there is \( a \in A \) such that \( \pi_s(\iota(f) a) = 0 \) and thus \( \iota(f) a \notin s \). On the other hand, \( \iota(f) a \in I \) as, for each \( s' \notin \psi^{-1}(U), \pi_{s'}(\iota(f) a) = f(\psi(s')) \pi_{s'}(a) = 0 \). It follows that \( I \notin s \) and therefore \( s \in U(I) \).

Conversely, suppose that \( \psi: \text{Prim}(A) \to X \) is a continuous map. Obviously we have a unital \( * \)-homomorphism \( \tau: C_0(X) \to C_0(\text{Prim}(A)) = \beta M(A) \) given by \( \tau(f) = f \circ \psi \). Define \( \iota := \tau|_{C_0(X)}: C_0(X) \to \beta M(A) \). Suppose there is \( t \in \text{Prim}(A) \) such that \( \iota(C_0(X)) A \subseteq t \). Let \( f \in C_0(X) \) such that \( f(\psi(t)) \neq 0 \). For all \( a \in A \), \( 0 = \iota(f)a + t = f(\psi(t))(a + t) \) implying that \( A \subseteq t \) which is impossible. It follows that \( \iota(C_0(X)) A = A \) as required.

**Remark 4.2** The above argument shows that a \( C_0(X) \)-algebra is nothing but a unital \( * \)-homomorphism \( C_0(X) \to C_0(\text{Prim}(A)) = C(\beta X) \to C_0(\text{Prim}(A)) = C(\beta \text{Prim}(A)) \) with the property that the image of \( \text{Prim}(A) \) under the canonical map

\[
\text{Prim}(A) \to \beta \text{Prim}(A) \to \beta X
\]

lands in \( X \subseteq \beta X \) (where “\( \beta \)” stands for Stone–Čech compactification).

5 From sheaves to \( C^* \)-bundles

Let \( \mathcal{A} \) be a presheaf of \( C^* \)-algebras over a topological space \( X \). We emphasize that \( X \) is in general not assumed to be Hausdorff. We want to associate to \( \mathcal{A} \) a canonical bundle of \( C^* \)-algebras; the following concept of \( C^* \)-bundle turns out to be the correct one in our context, see Remark 5.5 below.

**Definition 5.1** For a topological space \( X \), an upper semicontinuous \( C^* \)-bundle over \( X \) (in short, a usc \( C^* \)-bundle over \( X \)) is a triple \( (A, \pi, X) \) consisting of a topological space \( A \) and an open, continuous surjection \( \pi: A \to X \) with each fibre \( A_x := \pi^{-1}(x) \) a \( C^* \)-algebra and such that the function \( \| \cdot \|: A \to \mathbb{R} \) defined by \( a \mapsto \|a\|_{A_{\pi(a)}} \) is upper semicontinuous and all algebraic operations are continuous on \( A \); that is, \( + \) and \( \cdot \) are continuous functions \( A \times_A A \to A \) (where \( A \times_A A = \{(a_1, a_2) \in A \times A | \pi(a_1) = \pi(a_2)\} \)) and \( ^* : A \to A \) as well as \( - : C \times A \to A \) are continuous.

Denoting by \( \Gamma_0(U, A), U \in \mathcal{O}_X \) the set of all bounded continuous sections \( s: U \to A \) of \( \pi \) we further require the following properties.

(i) For all \( U \in \mathcal{O}_X \), \( s \in \Gamma_0(U, A) \) and \( \varepsilon > 0 \), the set

\[
V(U, s, \varepsilon) := \{a \in A | \pi(a) \in U \text{ and } \|a - s(\pi(a))\| < \varepsilon\}
\]

is an open subset of \( A \) and these sets form a basis for the topology of \( A \).

(ii) For each \( x \in X \), we have

\[
A_x = \{s(x) | s \in \Gamma_0(U, A), U \text{ an open neighbourhood of } x\}.
\]

We say the usc \( C^* \)-bundle is unital in case each \( A_x \) is a unital \( C^* \)-algebra. Often we will also speak of the total space \( A \) of the bundle as the bundle.

It is easy to see that the relative topology a fibre \( A_x \) inherits from the total space \( A \) in a usc \( C^* \)-bundle is nothing but the norm topology of the \( C^* \)-algebra \( A_x \). For instance, let \( x \in X \) and \( b \in A_x \). For \( \varepsilon > 0 \), there are \( U \in \mathcal{O}_X \) with \( x \in U \) and \( s \in \Gamma_0(U, A) \) such that \( \|s(x) - b\| < \frac{\varepsilon}{2} \), by condition (ii) above. The relative open subset

\[
V(U, s, \frac{\varepsilon}{2}) \cap A_x = \{z \in A_x | \|z - s(x)\| < \frac{\varepsilon}{2}\}
\]

is contained in the \( \varepsilon \)-ball about \( b \), \( \{a \in A_x | \|a - b\| < \varepsilon\} \). Therefore, the relative \( A \)-topology on \( A_x \) is stronger than the norm topology on \( A_x \). It is similarly straightforward to show the reverse inclusion between these topologies.
**Lemma 5.2** Let \( (A, \pi, X) \) be an upper semicontinuous \( C^* \)-bundle, and let \( Y \) be a subset of \( X \), endowed with the subspace topology. Then \( \Gamma_b(Y, A) \) is a \( C^* \)-algebra.

**Proof.** The set \( \Gamma_b(Y, A) \) carries natural algebraic operations and is endowed with the supremum norm; so all we need to check is completeness. Take a Cauchy sequence \( (s_n) \) in \( \Gamma_b(Y, A) \). There is a well-defined section \( s \) on \( Y \) defined by \( s(x) = \lim_{n \to \infty} s_n(x) \), for \( x \in Y \). To prove the continuity of \( s \), take \( x_0 \in Y \). Let \( V(U, t, \varepsilon) \) be a basic neighbourhood of \( s(x_0) \) in \( A \), so that \( \|s(x_0) - t(x_0)\| < \varepsilon \). Set \( \varepsilon' := \|s(x_0) - t(x_0)\| \) and \( \delta := \varepsilon - \varepsilon' \). There is a positive integer \( n_0 \) such that, for \( n \geq n_0 \), we have

\[
\|s(x) - s_n(x)\| < \delta/2 \quad \text{for all} \quad x \in Y.
\]

Using the triangle inequality we conclude that \( \|s_n(x_0) - t(x_0)\| < \delta/2 + \varepsilon' \) for all \( n \geq n_0 \). Since \( \cdot \|_A \) is upper semicontinuous, the map \( U \cap Y \to \mathbb{R} \) defined by \( x \mapsto \|(s_{n_0} - t)(x)\|_A \) is upper semicontinuous; hence, there is an open subset \( W \subseteq U \) such that

\[
W \cap Y = \{x \in U \cap Y \mid \|(s_{n_0} - t)(x)\| < \delta/2 + \varepsilon'\}
\]

and \( x_0 \in W \). For \( x \in W \cap Y \), we get

\[
\|s(x) - t(x)\| \leq \|s(x) - s_{n_0}(x)\| + \|s_{n_0}(x) - t(x)\| < \delta/2 + \delta/2 + \varepsilon' = \varepsilon.
\]

It follows that \( s(W \cap Y) \subseteq V(U, t, \varepsilon) \), which shows that \( s \) is continuous at \( x_0 \).

Since the convergence is uniform, it is evident that \( s \) is bounded. \( \square \)

**Theorem 5.3** Let \( (A, \pi, X) \) be an upper semicontinuous \( C^* \)-bundle. Then the assignment \( \mathfrak{A}(U) = \Gamma_b(U, A) \), where \( \Gamma_b(U, A) \) denotes the \( C^* \)-algebra of bounded continuous sections on \( U \in \mathcal{O}_X \) and \( \Gamma_b(\emptyset, A) = \{0\} \), defines a sheaf of \( C^* \)-algebras.

**Proof.** This follows from Lemma 5.2. \( \square \)

**Remark 5.4** Let \( (A, \pi, X) \) be a \( * \) \( C^* \)-bundle. For an open subset \( U \) of \( X \), set \( A_U = \bigsqcup_{x \in U} A_x \) with the induced topology; this is a \( * \) \( C^* \)-bundle over \( U \). However, if \( Y \) is an arbitrary subset of \( X \), then, although we can consider the sheaf \( \mathfrak{A}(U \cap Y) = \Gamma_b(U \cap Y, A) \), it is possible that the topology on \( A|_Y \) is different from the induced topology of \( A|_Y \).

**Remark 5.5** It is customary to work with bundles of \( C^* \)-algebras over a locally compact Hausdorff space \( X \). Traditionally, the norm function on the total space \( A \) is assumed to be continuous and one speaks of a continuous \( C^* \)-bundle or continuous field of \( C^* \)-algebras; see, e.g., [4] and [8]. More generally, upper semicontinuous bundles of \( C^* \)-algebras were studied in [9], [15], [16], [20] and used more recently in [19], for instance. In this setting, the topology on the total space is uniquely determined by canonical properties of the bundle; this is Fell’s theorem described in detail in [8, Theorem II.13.18] in the continuous case. A very nice exposition on the more general, upper semicontinuous case is contained in [26, Appendix C.2]. As a result, this topology on \( A \) is often not explicitly mentioned. It also turns out that there is a bijective correspondence between upper semicontinuous \( C^* \)-bundles over locally compact Hausdorff spaces and \( C_0(X) \)-algebras mentioned above in Section 4, [20, Theorems 2.3 and 3.3] or [26, Theorem C.26]. Note that, in this generality, \( A \) need not be Hausdorff even though \( X \) is [26, Example C.27].

In our extended setting of entirely arbitrary base spaces we find it more natural to include the topology into the definition of a \( * \) \( C^* \)-bundle, especially for the connection with sheaves.

We now want to associate to a presheaf \( \mathfrak{A} \) of \( C^* \)-algebras a \( * \) \( C^* \)-bundle \( (A, \pi, X) \). This is done similarly to the procedure in algebraic geometry using the \( \acute{e} \)tale space, but we have to take into account the right topology.
Theorem 5.6 Given a presheaf $\mathfrak{A}$ of $C^*$-algebras over $X$, there is a canonically associated upper semicontinuous $C^*$-bundle $(\pi, X, \mathcal{O})$ over $X$.

Proof. Let $\mathfrak{A}$ be a presheaf of $C^*$-algebras over $X$. For $x \in X$, define the stalk at $x$ by $A_x := \lim_{\to x} \mathfrak{A}(U)$ as the direct limit of $C^*$-algebras of the directed family $\{\mathfrak{A}(U)\}$, where $U$ ranges over the family of all open neighbourhoods of $x$ in $X$. Take $A := \bigsqcup_{x \in X} A_x$ and let $\pi(a) = a$ if $a \in A_x$. Then $a \in \mathfrak{A}(U)$ and $x \in U$, we have a canonical $*$-homomorphism $\mathfrak{A}(U) \to A_x$ and we denote by $s(x)$ the image of $s$ under this mapping.

For an open subset $U$ of $X$, an element $s \in \mathfrak{A}(U)$, and $\varepsilon > 0$, define

$$V(U, s, \varepsilon) = \{a \in A | \pi(a) \in U \text{ and } \|a - s(\pi(a))\| < \varepsilon\}.$$ 

We define a topology on $A$ by declaring the open subsets as those unions of subsets of the above form $V(U, s, \varepsilon)$. In order to show that this family defines a topology on $A$, we have to check that an intersection $V(U_1, s_1, \varepsilon_1) \cap V(U_2, s_2, \varepsilon_2)$ of two of these sets is a union of sets of the same form.

Assume that the intersection above is non-empty and take an element $a$ in the intersection. Then $x := \pi(a) \in U_1 \cap U_2$ and $\|a - s_1(x)\| < \varepsilon_1, \|a - s_2(x)\| < \varepsilon_2$. Choose $0 < \delta < \min\{\varepsilon_1 - \|a - s_1(x)\|, \varepsilon_2 - \|a - s_2(x)\|\}$.

Since $a \in A_x$, there are open neighbourhoods $W$ of $x$, with $W \subseteq U_1 \cap U_2$, such that $\|a - s(x)\| < \delta$. Note that

$$\|s(x) - s_1(x)\| \leq \|s(x) - a\| + \|a - s_1(x)\| < \delta/2 + \|a - s_1(x)\| < \varepsilon_1 - \delta/2,$$

and similarly $\|s(x) - s_2(x)\| < \varepsilon_2 - \delta/2$. Thus there is an open neighbourhood $W'$ of $x$, contained in $W$, such that $\|s_{W'} - s_{W}W\| < \varepsilon_1 - \delta/2$. Choose $0 < \delta < \min\{\varepsilon_1 - \|a - s_1(x)\|, \varepsilon_2 - \|a - s_2(x)\|\}$.

Hence $a \in V(W, s, \varepsilon)$ and, for $b \in V(W, s_{W}W, \varepsilon)$, we have

$$\|b\| \leq \|s_{W}w\| + \|b - s(\pi(b))\| < \|a\| + \|s(x)\| + \|s_{W}W\| + \varepsilon < \|a\| + 3\varepsilon = \alpha;$$

thus $V(W, s_{W}W, \varepsilon) \subseteq \{c \in A | \|c\| < \alpha\}$, as desired.

It is straightforward (though tedious) to show that all the algebraic operations on $A$ are continuous. It is a simple exercise to show that $A$ is open and continuous. Observe that, for any open subset $U$ of $X$, elements of $\mathfrak{A}(U)$ give rise to bounded continuous sections on $U$ via $\mathfrak{A}(U) \to \Gamma_b(U, \mathcal{O})$, $s \to s(x)$ for $x \in U$.

To check the remaining requirements in Definition 5.1, let $s \in \Gamma_b(U, A)$ where $U$ is an open subset of $X$ and let $\varepsilon > 0$. We need to show that $V(U, s, \varepsilon)$ is a union of basic open sets for the topology of $A$. Indeed, for $a \in V(U, s, \varepsilon)$, there is $s' \in \mathfrak{A}(U')$ with $\|s'(\pi(a)) - a\| < \delta$ for some open neighbourhood $U'$ of $\pi(a)$ contained in $U$, where $2\delta = \varepsilon - \|a - s(\pi(a))\| > 0$. Letting $x = \pi(a)$ we have $\|s(x) - s'(x)\| < \varepsilon - \delta$ so that $s(x) \in V(U', s', \varepsilon - \delta)$. Using the continuity of $s$, we can find a smaller open neighbourhood $W \subseteq U'$ of $x$ such that $s(y) \in V(U', s', \varepsilon - \delta)$ for all $y \in W$. It follows that, for $b \in V(W, s', \delta)$, 

$$\|b - s(\pi(b))\| \leq \|b - s'(\pi(b))\| + \|s'(\pi(b)) - s(\pi(b))\| < \delta + (\varepsilon - \delta) = \varepsilon,$$

showing that $a \in V(W, s', \delta) \subseteq V(U, s, \varepsilon)$, as desired.

Finally, condition (ii) in Definition 5.1 is satisfied by the very construction of $A_x, x \in X$. 

Remark 5.7 Note that if in Theorem 5.6, $\mathfrak{A}$: $\mathcal{O}_X \to C^*_+$ then the bundle $\mathcal{A}$ consists of unital $C^*$-algebras.

It is easy to check that if we start out with a $C^*$-bundle $(\mathcal{A}, \pi, X)$, consider the sheaf $\mathfrak{A}(U) = \Gamma_b(U, \mathcal{A})$, $U \in \mathcal{O}_X$ and then construct the $C^*$-bundle associated with $\mathfrak{A}$ as in Theorem 5.6, we get back the original bundle $\mathcal{A}$.
In the reverse direction, we start from a sheaf \( \mathfrak{A} \). It is clear that, for each \( U \in \mathscr{O}_X \), we have a \(*\)-homomorphism
\[
\mu_U : \mathfrak{A}(U) \to \Gamma_b(U, A) \text{ satisfying } \mu_U(s)(x) = s(x), \quad x \in U, \tag{5.2}
\]
into the sections of the bundle \( A \) constructed in Theorem 5.6, which is injective. Indeed, assume that \( \mu_U(s) = 0 \) for some \( s \in \mathfrak{A}(U) \). Given \( \varepsilon > 0 \) and \( x \in U \), there exists an open subset \( W_x \) of \( U \) such that \( x \in W_x \) and \( \|s|_{W_x}\| < \varepsilon \). Since \( \mathfrak{A} \) is a sheaf, we have an injective \(*\)-homomorphism \( \mathfrak{A}(U) \subseteq \prod_{x \in U} \mathfrak{A}(W_x) \), from which we conclude that \( \|s\| \leq \varepsilon \). Since \( \varepsilon \) is arbitrary, we get \( s = 0 \).

The question whether the map \( \mu_U : \mathfrak{A}(U) \to \Gamma_b(U, A) \) must be surjective seems to be more difficult, and, a priori, there appears to be no good reason for this to happen. A useful observation here is that \( \Gamma_b(U, A) \) is a \( C_b(U) \)-algebra, thus, if the map \( \mu_U \) is an isomorphism then \( \mathfrak{A}(U) \), too, must be a \( C_b(U) \)-module.

**Definition 5.8** Let \( \mathfrak{C} \) be a sheaf of commutative unital \( C^* \)-algebras over a topological space \( X \). We say a sheaf \( \mathfrak{A} \) of unital \( C^* \)-algebras is a \( \mathfrak{C} \)-sheaf if, for every \( U \in \mathscr{O}_X \), the \( C^* \)-algebra \( \mathfrak{A}(U) \) is a unital \( \mathfrak{C}(U) \)-Banach module and, whenever \( V \subseteq U \), the restriction map \( \mathfrak{A}(U) \to \mathfrak{A}(V) \) is a module homomorphism; that is, \((ha)_V = (h|_V)(a|_V)\) for all \( a \in \mathfrak{A}(U) \) and \( h \in \mathfrak{C}(U) \).

**Definition 5.9** Let \( X \) be a topological space. The sheaf \( \mathfrak{C}(X) \) of unital \( C^* \)-algebras over \( X \) is given by \( \mathfrak{C}(U) = C_b(U), \ U \in \mathscr{O}_X \), and the evident restriction mappings.

Therefore, if \( X \) is locally compact Hausdorff, \( \mathfrak{C}(X) \) is nothing but the multiplier sheaf over \( X \).

Suppose that \( \mathfrak{A} \) is a \( \mathfrak{C}(X) \)-sheaf. Note that \( C_b(U) = C(\beta U) \) and, for \( V \subseteq U \), we have \( (ha)_V = (h|_V)(a|_V) \), whenever \( h \in C_b(U) \) and \( a \in \mathfrak{A}(U) \). This yields a \( C_b(U) \)-module homomorphism \( \mu_U : \mathfrak{A}(U) \to \Gamma_b(U, A) \). We have to show that \( (h \cdot a)(x) = h(x)a(x) \) for every \( x \in U \). Let \( \varepsilon > 0 \). By the continuity of \( h \), there exists a smaller neighbourhood \( W \) of \( x \) such that \( \|h|_W - h(x)\| < \varepsilon \). Thus we have
\[
\|(ha)_V - h(x)a|_W\| = \|h|_W \cdot a|_W - h(x)a|_W\| \leq \varepsilon \|a\|,
\]
which implies that \((ha)(x) - h(x)a(x) = 0\).

The condition of being a \( \mathfrak{C}(X) \)-sheaf is satisfied by our main examples discussed in Section 3. Consider a \( C^* \)-algebra \( A \) and the multiplier sheaf \( \mathfrak{M}_A \) over \( \text{Prim}(A) \). For \( U \in \mathfrak{C}(\text{Prim}(A)) \), we have \( C_b(U) \cong ZM(A(U)) \) and thus \( \mathfrak{M}_A(U) = M(A(U)) \) is a \( C_b(U) \)-module. The compatibility condition is obvious. Now suppose that \( \mathfrak{A} \) is a sheaf over \( Y \) satisfying this condition and \( \psi : Y \to X \) is a continuous map. Then \( \psi_* (\mathfrak{A}) \) also satisfies this condition because the map \( \psi^* : C_b(U) \to C_b(\psi^{-1}(U)) \) allows us to define a suitable \( C_b(U) \)-module structure on \( \psi_* (\mathfrak{A})(U) = \mathfrak{A}(\psi^{-1}(U)) \) for \( U \in \mathscr{O}_X \). Consequently, if \( A \) is a \( C^* \)-algebra over \( X \), then \( \psi_* (\mathfrak{M}_A) \) is a \( \mathfrak{C}(X) \)-sheaf.

Consider next the injective envelope sheaf of \( A \). For each open subset \( U \) of \( \text{Prim}(A) \), we have an injective \(*\)-homomorphism
\[
C_b(U) \cong ZM(A(U)) \longrightarrow Z(I(A(U))) = \mu_U Z(I(A)),
\]
compare [13]; this gives the desired structure on the sheaf \( \mathfrak{I}_A \).

We shall now discuss two settings in which the mappings \( \mu_U \) in (5.2) are in fact isomorphisms; see also Corollary 6.11 below.

**Proposition 5.10** Let \( X \) be a second countable, locally compact Hausdorff space. Let \( \mathfrak{A} \) be a \( \mathfrak{C}(X) \)-sheaf of unital \( C^* \)-algebras over \( X \). Then the natural maps \( \mu_U : \mathfrak{A}(U) \to \Gamma_b(U, A) \) are isomorphisms for all \( U \in \mathscr{O}_X \).

**Proof.** Let \( U \) be an open subset of \( X \), and take \( s \in \Gamma_b(U, A) \). Then \( U \) is a paracompact space, see, e.g., [22, Proposition 1.7.11]. Let \( x \in U \) and \( \varepsilon > 0 \). Then, by the definition of \( A_x \), there are open neighbourhoods \( W_x \subseteq U \) of \( x \) and \( a_x \in \mathfrak{A}(W_x) \) such that \( \|s|_{W_x}(a_x) - s|_{W_x}\| < \varepsilon \). Since \( U \) is paracompact, we get a locally finite refinement \( U = \bigcup_i U_i \) and \( a_i \in \mathfrak{A}(U_i) \) such that \( \|s|_{U_i}(a_i) - s|_{U_i}\| < \varepsilon \) for each \( i \). Let \( (h_i) \) be a partition of unity subordinated to the open covering \((U_i)\) of \( U \), that is, \( 0 \leq h_i \leq 1 \), \( F_i := \{x \in U : h_i(x) \neq 0\} \subseteq U_i \) for all \( i \) and \( \sum h_i = 1 \). We are going to define an element \( a = \sum_i h_i a_i \in \mathfrak{A}(U) \). First note that there is a well-defined element \( h_i a_i \) in \( \mathfrak{A}(U) \) extending \( h_i a_i \in \mathfrak{A}(U_i) \). Indeed, using the open covering \( U = U_i \cup (U \setminus F_i) \) and the sheaf property we obtain a unique element \( h_i a_i \in \mathfrak{A}(U) \) such that \( h_i a_i \big|_{U_i} = h_i a_i \) and \( h_i a_i \big|_{U \setminus F_i} = 0 \).
For each \( x \in U \), there is an open neighbourhood \( V_x \) of \( x \) such that \( V_x \) cuts only a finite number of sets \( U_i \), since the covering \( \{ U_i \} \) is locally finite. Let \( G_x \) be the finite set of the indices \( i \) such that \( V_x \cap U_i \neq \emptyset \). We define \( s_{V_x} \in \mathfrak{A}(V_x) \) by \( s_{V_x} = \sum_{i \in G_x} \left( h_i a_i \right) \). Clearly, this gives a compatible family \( \{ s_{V_x} \} \), and by the sheaf property of \( \mathfrak{A} \), there is a unique element \( a = \sum h_i a_i \in \mathfrak{A}(U) \) with \( a|_{V_x} = s_{V_x} \). For \( x \in U \), let \( J_x \subseteq G_x \) be the finite family of indices \( j \) such that \( x \in U_j \). We have

\[
\| s(x) - \mu_U(a)(x) \| \leq \sum_{j \in J_x} h_j(x) \left\| s(x) - \mu_{V_x \cap U_j} (a|_{V_x \cap U_j})(x) \right\| < \varepsilon.
\]

Therefore, the image of \( \mu_U \) is dense in the \( \text{C}^* \)-algebra \( \Gamma_b(U, A) \) but since it is also closed, we find that \( \mu_U \) is surjective.

\[\square\]

**Remark 5.11** Let \( \mathfrak{A} \) be a \( C(X) \)-sheaf of unital \( \text{C}^* \)-algebras over a compact Hausdorff space \( X \). In this case, the partition of unity argument in the above proof works for the space \( X \) so that \( \mathfrak{A}(X) \cong \Gamma_b(X, A) \) via the mapping \( \mu_X \).

Another context in which we can recover the sheaf from the \( \text{C}^* \)-bundle is that of Alexandrov spaces. These spaces are attached to a preordered set \( (X, \leq) \). We can recover the preordered set \( (X, \leq) \) for the Alexandrov topology via the specialisation order, defined by \( x \leq y \) if and only if the closure of \( \{ x \} \) is contained in the closure of \( \{ y \} \).

By [18, Lemma 2.32], \( X \) is an Alexandrov space if and only if every point \( x \) in \( X \) has a smallest neighbourhood \( U_x \). In particular, all finite topological spaces are Alexandrov spaces. Alexandrov spaces are in general highly non-Hausdorff. See [18, Section 2] for a discussion on such spaces in connection with the theory of \( \text{C}^* \)-algebras over topological spaces.

Observe that, by the gluing property of sheaves, a sheaf \( \mathfrak{A} \) of \( \text{C}^* \)-algebras over an Alexandrov space is determined by the \( \text{C}^* \)-algebras \( \mathfrak{A}(U_x) \) and the restriction maps \( \mathfrak{A}(U_x) \to \mathfrak{A}(U_y) \) for \( y \in U_x \). An easy case to have in mind is the one corresponding to the partially ordered set \( (X, \leq) \) with \( X = \{ x_1, x_2, x_3, x_4 \} \) and \( x_1 \leq x_2, x_3 \leq x_4 \). In this case, setting \( U_i = U_{x_i} \), we have \( U_1 = \{ x_4 \}, U_3 = \{ x_3, x_4 \}, U_2 = \{ x_2, x_4 \} \) and \( U_1 = X \). A sheaf of \( \text{C}^* \)-algebras over this space \( X \) is just a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{A}(U_1) & \to & \mathfrak{A}(U_2) \\
\downarrow & & \downarrow \\
\mathfrak{A}(U_3) & \to & \mathfrak{A}(U_4)
\end{array}
\]

The \( \text{C}^* \)-algebra \( \mathfrak{A}(U) \) corresponding to the open subset \( U = U_2 \cup U_3 \) is the pullback of the diagram \( \mathfrak{A}(U_2) \to \mathfrak{A}(U_3) \) and the map \( \mathfrak{A}(U_1) \to \mathfrak{A}(U) \) is obtained from the pullback property. So (5.3) determines completely the structure of the sheaf.

**Proposition 5.12** Let \( \mathfrak{A} \) be a sheaf of \( \text{C}^* \)-algebras over an Alexandrov space \( X \). Then the natural map \( \mu_U : \mathfrak{A}(U) \to \Gamma_b(U, A) \) is an isomorphism for every open subset \( U \) of \( X \).

**Proof.** By the gluing property, it suffices to check the statement for the minimal open neighbourhoods \( U_x \), \( x \in X \). Assume that \( s \in \Gamma_b(U_x, A) \) for some \( x \in X \). Observe that \( \mathfrak{A}_x = \mathfrak{A}(U_x) \), so that \( s(x) \in \mathfrak{A}(U_x) \). We will show that \( s = \mu_{U_x}(s(x)) \). For \( \varepsilon > 0 \), consider the open neighbourhood \( V(U_x, \mu_{U_x}(s(x)), \varepsilon) \) of \( s(x) \) in \( \mathfrak{A} \). The continuity of \( s \) and the minimality of \( U_x \) imply that \( s(U_x) \subseteq V(U, \mu_{U_x}(s(x)), \varepsilon) \). Therefore, for all \( y \in U_x \), we have \( \| s(y) - \mu_{U_x}(s(x))(y) \| < \varepsilon \). Since this holds for every \( \varepsilon > 0 \), we conclude that \( s = \mu_{U_x}(s(x)) \). \[\square\]
6 From $C^*$-bundles to sheaves

In this section we aim to show that, for a separable unital $C^*$-algebra $A$ with Hausdorff primitive spectrum, the upper semicontinuous $C^*$-bundle traditionally associated to $A$ (see, e.g., [9] or [20]) agrees with the one obtained from our canonical multiplier sheaf above; cf. Theorem 5.6.

Let $(A, \psi)$ be a $C^*$-algebra over a topological space $X$, and let $\mathfrak{M}_X = \psi_*(\mathfrak{M}_A)$ be the multiplier sheaf associated to $(A, \psi)$, see Remark 3.7. Let $(A, \pi_X)$ be the usc $C^*$-bundle associated to $\mathfrak{M}_X$ via Theorem 5.6. Thinking of the elements in $A_x$, $x \in X$ as germs of functions at $x$, we shall define a mapping into a certain $C^*$-algebra $A_x$ acting like evaluation at $x$. It turns out that, in the presence of good local conditions, this map $A_x \rightarrow A_x$ is an isomorphism, so that germs are determined by their values at $x$; compare with the commutative case in the Introduction.

**Definition 6.1** Let $\mathfrak{M}_X = \psi_*(\mathfrak{M}_A)$ be the multiplier sheaf associated to the $C^*$-algebra $(A, \psi)$ over $X$. Define

$$A_x := \lim_{x \in U} M(A(U) + I_x/I_x),$$

(6.1)

where $U$ ranges over all open neighbourhoods of $x$ in $X$, and $I_x := A(x \setminus \{x\})$. (Recall that $A(U) = A(\psi^{-1}(U))$ designates the closed ideal of $A$ determined by $\psi^{-1}(U) \in \mathcal{O}_{\text{Prim}(A)}$.)

Observe that in the situation of the $C^*$-algebra $(A, \text{id}_{\text{Prim}(A)})$ over Prim$(A)$, we have $I_t = t$ and $A_t = M_{\text{loc}}(A/t)$, the local multiplier algebra of the primitive $C^*$-algebra $A/t$, for each $t \in \text{Prim}(A)$.

**Proposition 6.2** Let $\mathfrak{M}_X = \psi_*(\mathfrak{M}_A)$ be the multiplier sheaf associated to the $C^*$-algebra $(A, \psi)$ over $X$ and, for $x \in X$, let $A_x$ be the fibre over $x$ associated to it. Then there is a canonical *-homomorphism $\varphi_x : A_x \rightarrow A_x$, where $A_x$ is the $C^*$-algebra defined in Definition 6.1. This map is surjective if $A$ is a separable $C^*$-algebra.

**Proof.** Let $U \in \mathcal{O}_X$ be an open neighbourhood of $x$. Then we have a *-homomorphism $M(A(U)) \rightarrow M(A(U)/A(U) \cap I_x)$, which is surjective in case $A$ is separable [21, 3.12.10]. Thus we get a map

$$M(A(U)) \rightarrow M(A(U)/A(U) \cap I_x) \xrightarrow{\cong} M(A(U) + I_x/I_x) \rightarrow A_x$$

that is compatible with the restriction maps $M(A(U)) \rightarrow M(A(V))$ for $x \in V \subseteq U$. This results in a *-homomorphism $\varphi_x : A_x \rightarrow A_x$, which is surjective when $A$ is separable.

In particular, for the $C^*$-algebra $(A, \text{id}_{\text{Prim}(A)})$ over Prim$(A)$, we get a *-homomorphism $\varphi_t : A_t \rightarrow M_{\text{loc}}(A/t)$, for $t \in \text{Prim}(A)$.

Recall that the Jacobson topology on Prim$(A)$ is the coarsest topology on Prim$(A)$ such that the function $N(a) : \text{Prim}(A) \rightarrow \mathbb{R}, t \mapsto \|a + t\|$ is lower semicontinuous for each $a \in A$.

**Definition 6.3** Let $X$ be a topological space and $x \in X$. We say that $x$ is a separated point in $X$ if, for every $y$ not in the closure of $\{x\}$, $x$ and $y$ admit disjoint neighbourhoods.

We include the proof of the following well-known result, compare [7, 3.9.4].

**Lemma 6.4** For a $C^*$-algebra $A$, $t \in \text{Prim}(A)$ is a separated point if and only if all the maps $N(a), a \in A$, are continuous at $t$.

**Proof.** Assume first that all the maps $N(a)$ are continuous at $t$. Let $s \in \text{Prim}(A) \setminus \{t\}$. Since

$$\overline{\{t\}} = \{t' \in \text{Prim}(A) \mid t \subseteq t'\},$$

we have $t \nsubseteq s$. Taking $a \in t \setminus s$ we obtain

$$N(a)(t) = \|a + t\| = 0 \quad \text{and} \quad N(a)(s) = \|a + s\| \neq 0.$$

It follows that $t$ and $s$ can be separated by disjoint open subsets of $X$. 

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Now assume that \( t \) is a separated point in \( \text{Prim}(A) \) and take \( a \in A \). Since \( N(a) \) is lower semicontinuous, it suffices to show that, for \( \alpha > N(a)(t) \), there is an open neighbourhood \( V \) of \( t \) such that \( N(a)(V) \subseteq [0, \alpha) \). Given such \( \alpha \), consider the compact subset

\[
K = \{ s \in \text{Prim}(A) \mid \|a + s\| \geq \alpha \}.
\]

Let \( s \in K \); then \( s \notin \overline{\{t\}} \), because the set

\[
U = \{ t' \in \text{Prim}(A) \mid \|a + t'\| > \|a + t\| \}
\]

is open in \( \text{Prim}(A) \) and \( s \in U \) but \( t \notin U \). Since \( t \) is a separated point in \( \text{Prim}(A) \), it follows that there are disjoint open subsets \( U(s) \) and \( V(s) \) such that \( s \in U(s) \) and \( t \in V(s) \). By using the compactness of \( K \), we get an open neighbourhood \( V \) of \( t \) such that \( V \cap K = \emptyset \). If follows that \( N(a)(V) \subseteq [0, \alpha) \), as desired. \( \square \)

If \( A \) is separable, there is a dense \( G_\delta \) subset of \( \text{Prim}(A) \) consisting of separated points [7, 3.9.4]. Evidently, the set \( \text{Sep}(A) \) of separated points of \( \text{Prim}(A) \) consists of minimal elements in \( (\text{Prim}(A), \subseteq) \): if \( t_1 \subseteq t_2 \) then \( t_2 \in \overline{\{t_1\}} \) and therefore \( t_1 \) and \( t_2 \) cannot have disjoint neighbourhoods. If \( t_2 \in \text{Sep}(A) \), this implies that \( t_1 \in \overline{\{t_2\}} \), i.e., \( t_2 \subseteq t_1 \).

**Proposition 6.5** Let \( A \) be a \( C^* \)-algebra and let \( t \in \text{Prim}(A) \). Then \( t \in \text{Sep}(A) \) if and only if the kernel of the natural map \( \lambda : A_t \to M_{\text{loc}}(A/t) \) is trivial.

**Proof.** Since the kernel of the composition \( \lambda : A_t \to M_{\text{loc}}(A/t) \) is trivial, we find that the kernel of the natural map \( \lambda : A_t \to A \) is always contained in \( t \). Recall that \( a_{U} \) denotes the image of \( a \in A \) under the canonical \(*\)-homomorphism \( A \to M(A(U)) \) so that \( \|a_{U}\| = \sup\{\|a + t'\| \mid t' \in U \} \).

Assume that \( t \) is a separated point in \( \text{Prim}(A) \) and let \( a \) be an element in \( t \). Since \( N(a)(t) = 0 \) and \( N(a) \) is continuous at \( t \) by Lemma 6.4, it follows that, for every \( \varepsilon > 0 \), there is an open neighbourhood \( U \) of \( t \) such that \( N(a)(s) < \varepsilon \) for all \( s \in U \). We conclude that \( \|a_{U}\| \leq \varepsilon \) and thus \( [a] = 0 \) in \( A_t \).

Conversely, assume that the kernel of the natural map \( \lambda : A_t \to A_t \) is trivial. If \( s \in \text{Prim}(A) \) and \( s \notin \overline{\{t\}} \), then \( t \notin s \). Take \( a \in t \setminus s \) with \( \|a + s\| = 1 \). By hypothesis, \( a \) belongs to the kernel of the natural map \( \lambda : A_t \). Therefore there is an open neighbourhood \( V \) of \( t \) such that \( N(a)(t') < 1/2 \) for every \( t' \in V \). On the other hand, since \( N(a) \) is lower semi-continuous, the set \( V = \{ s' \in \text{Prim}(A) \mid N(a)(s') > 1/2 \} \) is open in \( \text{Prim}(A) \) and it contains \( s \). Consequently, \( t \) and \( s \) can be separated by disjoint open subsets in \( \text{Prim}(A) \), and hence \( t \in \text{Sep}(A) \). \( \square \)

We are ready to provide a characterization of the points \( t \) in \( \text{Prim}(A) \) such that \( \varphi_t \) is injective.

**Theorem 6.6** Let \( A \) be a \( C^* \)-algebra and let \( t \in \text{Prim}(A) \). Then the natural map \( \varphi_t : A_t \to M_{\text{loc}}(A/t) \) is injective if and only if, for every open neighbourhood \( U \) of \( t \) in \( \text{Prim}(A) \), \( t \) is a separated point in \( \text{Prim}(M(A(U))) \).

**Proof.** We fix the following notation. For \( t \in \text{Prim}(A) \), let \( \pi_t : A \to B(H_t) \) be an irreducible representation such that \( \text{ker} \pi_t = t \). If \( t \in U \in \mathcal{O}(\text{Prim}(A)) \), then the restriction of \( \pi_t \) to \( A(U) \) gives an irreducible representation of \( A(U) \) on the same Hilbert space \( H_t \). The unique extension of \( \pi_t \) to a representation of \( M(A(U)) \) on \( H_t \) will be denoted by \( \varphi_t \). (We omit the reference to \( U \) to simplify the notation.) Then \( U \) is considered as a dense open subset of \( \text{Prim}(M(A(U))) \) via the correspondences \( t \mapsto t \cap U \mapsto \text{ker} \varphi_t \).

Note that, by Proposition 6.5, \( t \) is a separated point in \( \text{Prim}(M(A(U))) \) if and only if the kernel of the canonical map \( \lambda_{U} : M(A(U)) \to A_t \) is precisely \( \text{ker} \varphi_t \). On the other hand, \( \varphi_t : A_t \to M_{\text{loc}}(A/t) \) is injective if and only if the restriction of \( \varphi_t \) to \( \lambda_U(M(A(U))) \) is injective for every open neighbourhood \( U \) of \( t \) in \( \text{Prim}(A) \).

Thus the claim is proved once we can show that the restriction of \( \varphi_t \) to \( \lambda_U(M(A(U))) \) is injective if and only if \( \text{ker} \lambda_U = \text{ker} \varphi_t \).

In the commutative diagram

\[
\begin{array}{ccc}
M(A(U)) & \xrightarrow{\lambda_U} & A_t \\
\downarrow & & \downarrow \varphi_t \\
M(A(U) + t/t) & \xrightarrow{\varphi_t} & M_{\text{loc}}(A/t)
\end{array}
\]

(6.2)

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the kernel of the natural map \( M(A(U)) \to M(A(U) + t/t) \) is precisely \( \ker \varpi_t \), and the map \( M(A(U) + t/t) \to M_{\text{loc}}(A/t) \) is injective. From this we conclude that the inclusion \( \ker \lambda_U \subseteq \ker \varpi_t \) always holds, and that the equality \( \ker \lambda_U = \ker \varpi_t \) holds if and only if the restriction of \( \varphi_t \) to \( \lambda_U(M(A(U))) \) is injective.

This concludes the proof.

Since the map \( \varphi_t \) is always surjective when \( A \) is a separable \( C^* \)-algebra, we obtain the following consequence.

**Corollary 6.7** Let \( A \) be a separable \( C^* \)-algebra and let \( t \in \text{Prim}(A) \). Then the natural map \( \varphi_t: A_t \to M_{\text{loc}}(A/t) \) is an isomorphism if and only if, for every open neighbourhood \( U \) of \( t \) in \( \text{Prim}(A) \), \( t \) is a separated point in \( \text{Prim}(M(A(U))) \).

**Remark 6.8** The following example was communicated to us by Douglas Somerset. Let \( A \) be a separable \( C^* \)-algebra of sequences of complex matrices converging to a scalar multiple of \( e_{11} \) (so that all entries in the limit are zero but for the \( (1,1) \)-entry). Then \( A \) is a separable \( C^* \)-algebra with Hausdorff primitive ideal space but the maximal ideal \( t \) of \( A \) corresponding to the evaluation at the limit of the \( (1,1) \)-entries is not separated in \( \text{Prim}(M(A)) \). This shows that the hypothesis in Theorem 6.6 cannot be relaxed in general.

We will now specialise to points in \( \text{Prim}(A) \) which are both separated and closed.

**Lemma 6.9** Let \( A \) be a unital \( C^* \)-algebra, and assume that \( t \in \text{Sep}(A) \) is a closed point in \( \text{Prim}(A) \). Denote by \( A_t \) the fibre over \( t \) corresponding to the multiplier sheaf of \( A \). Then the natural map \( \varphi_t: A_t \to A/t \) is an isomorphism.

**Proof.** Since \( A \) is unital and \( t \) is a closed point in \( \text{Prim}(A) \), the \( C^* \)-algebra \( A/t \) is simple and unital so that \( M_{\text{loc}}(A/t) = A/t \). The map \( \varphi_t: A_t \to M_{\text{loc}}(A/t) \) given by Proposition 6.2 therefore simplifies to \( \varphi_t: A_t \to A/t \). The composition \( A \to A_t \to A/t \) of \( \varphi_t \) with the canonical \( * \)-homomorphism \( A \to A_t \) is just the canonical quotient map \( A \to A/t \). By Proposition 6.5, the kernel of the map \( A \to A_t \) is \( t \), so that we get a section of \( \varphi_t \). Consequently, \( \varphi_t \) is injective when restricted to the image of \( A \) in \( A_t \).

Let \( U \) be an open neighbourhood of \( t \) in \( \text{Prim}(A) \), and take \( m \in M(A(U)) \). Choose \( e \in A(U)_+ \) with the property that \( \|e\| = 1 \) and \( e + t = 1 + t \) in \( A/t \). (Note that \( A(U) + t/t = A/t \) as \( A/t \) is simple.) Observe that \( N(1 - e)(t) = 0 \). Since \( t \) is a separated point, it follows from Lemma 6.4 that \( N(1 - e) \) is continuous at \( t \). Hence there is an open neighbourhood \( U_1 \) of \( t \) contained in \( U \) such that \( N(1 - e)(s) < 1/2 \) for every \( s \in U_1 \). Set \( Y = U_1 \), a closed subset of \( \text{Prim}(A) \), and consider \( U_2 = \text{Prim}(A) \setminus Y \), an open subset of \( \text{Prim}(A) \) with \( U_1 \cap U_2 = \emptyset \). Then \( A/A(U_2) \) is a unital \( C^* \)-algebra with primitive spectrum \( Y \) and \( A(U_1) \) sits as an essential ideal in it. It thus follows that we have an embedding of unital \( C^* \)-algebras \( A/A(U_2) \subseteq M(A(U_1)) = \mathfrak{M}_A(U_1) \). The set \( \{s \in \text{Prim}(A) \mid N(1 - e)(s) \leq 1/2\} \) is closed in \( \text{Prim}(A) \) and contains \( U_1 \); consequently \( N(1 - e)(s) \leq 1/2 \) for every \( s \in Y \). Since \( N_{A/(A(U_2))(1-e+A(U_2))(s)} = N_{A(1-e)(s)} \leq 1/2 \) for every \( s \in Y \), we get that \( \|1-e+A(U_2)\| \leq 1/2 < 1 \), and \( e + A(U_2) \) is invertible in \( A/A(U_2) \). Take any \( y \in A \) such that \( y + A(U_2) = (e + A(U_2))^{-1} \).

Then we have

\[
m_{\mathfrak{M}_A(U_1)}(m_{\mathfrak{M}_A(U_2)}(e + A(U_2))(y + A(U_2))) = (me + A(U_2))(y + A(U_2)) \in A/A(U_2),
\]

which shows that \( m_{\mathfrak{M}_A(U_2)} \) belongs to the image of the map \( A \to \mathfrak{M}_A(U_1) \). Thus we find that the image of \( m \) in \( A_t \) is \( \lim_{t \in \text{Prim}(A)} \mathfrak{M}_A(W) \) belongs to the image of the map \( A \to A_t \), and it turns out that the map \( A/t \to A_t \) is surjective. Since it is also injective, we conclude that it is an isomorphism, and so its inverse, \( \varphi_t \), must be an isomorphism too.

We now come to the main result in this section.

**Theorem 6.10** Let \( A \) be a unital \( C^* \)-algebra with Hausdorff primitive spectrum \( \text{Prim}(A) \). Then all the fibres \( A_t = A/t, t \in \text{Prim}(A) \) are isomorphic to the fibres \( A_t \) associated to the multiplier sheaf \( \mathfrak{M}_A \) of \( A \). Moreover, we have \( A \cong \Gamma(\text{Prim}(A), A) \) and \( A(U) \cong \Gamma(U, A) \) for each \( U \in \mathcal{O}_\text{Prim}(A) \).

**Proof.** Since every point in \( \text{Prim}(A) \) is closed and separated, we conclude from Lemma 6.9 that all the maps \( \varphi_t: A_t \to A_t, t \in \text{Prim}(A) \), are isomorphisms.

By Proposition 5.10 and Remark 5.11, we have

\[
A = \mathfrak{M}_A(\text{Prim}(A)) = \Gamma(\text{Prim}(A), A) = \Gamma(\text{Prim}(A), A).
\]
Finally, for each $U \in \mathcal{O}_{\text{Prim}(A)}$, we obtain

$$A(U) = C_0(U)A = C_0(U)\Gamma(\text{Prim}(A), A) = \Gamma_0(U, A).$$

The above theorem yields an isomorphism of $C^*$-bundles. We can indeed derive from it an isomorphism of sheaves, and thus obtain another instance where the sheaf of bounded continuous local sections agrees with the sheaf we start from.

**Corollary 6.11** Let $A$ be a unital $C^*$-algebra such that $\text{Prim}(A)$ is Hausdorff. Then the multiplier sheaf $\mathcal{M}_A$ of $A$ is isomorphic to the sheaf $\Gamma_b(-, A)$ of bounded continuous local sections of the $C^*$-bundle $A$ associated to $\mathcal{M}_A$.

**Proof.** Let $U \in \mathcal{O}_{\text{Prim}(A)}$. In the commutative diagram

$$
\begin{array}{ccc}
M(A(U)) & \xrightarrow{\mu_U} & \Gamma_b(U, A) \\
\downarrow & & \downarrow \\
A(U) & \xrightarrow{\mu_U} & \Gamma_0(U, A)
\end{array}
$$

$\mu_U$ is injective on $M(A(U))$ by (5.2) while, restricted to $A(U)$, it is an isomorphism by Theorem 6.10. The composition $\nu \circ \mu_U$ extends the isomorphism $\mu_U : A(U) \to \Gamma_0(U, A)$, so by commutativity of the diagram, must be an isomorphism. Since $\Gamma_0(U, A)$ is an essential ideal in $\Gamma_b(U, A)$, the mapping $\nu$ is injective. As $\nu$ is clearly surjective, it follows that $\mu_U : M(A(U)) \to \Gamma_b(U, A)$ is surjective as well.

The restriction mappings in the two sheaves are compatible with the isomorphisms $\mu_U$, from which we conclude that $\mathcal{M}_A \cong \Gamma_b(-, A)$.

**Remark 6.12** Under the conditions of Theorem 6.10, the norm function $A \to \mathbb{R}$ is continuous. This follows easily from the lower semicontinuity of the maps $N(a)$ for $a \in A$. Indeed, if $a + t \in A/t$ and $N(a)(t) > \alpha$ for some $\alpha \in \mathbb{R}_+$, then there is an open neighbourhood $U$ of $t$ in $\text{Prim}(A)$ such that $N(a)(t') > \alpha + \varepsilon$ for all $t' \in U$, for some $\varepsilon > 0$, and thus $\|b + t'\| > \alpha$ for every $b + t' \in V(U, a_U, \varepsilon)$. This shows that the norm function $\|\cdot\| : A \to \mathbb{R}$ is lower-semicontinuous, and thus continuous. We can recover the continuity of the norm functions $N(a)$, for $a \in A$, just by looking at the composition $\text{Prim}(A) \to A \to \mathbb{R}$ given by $t \mapsto a + t \mapsto \|a + t\|$. Moreover, all the functions $N(z) : U \to \mathbb{R}$, $N(z)(t) = \|z + t\|$, with $z \in M(A(U))$ are continuous, for $U \in \mathcal{O}_{\text{Prim}(A)}$ (regarding $U$ as an open subset of $\text{Prim}(M(A(U)))$).

**Remark 6.13** In general, the map $\varphi_t : A_t \to M_{\text{loc}}(A/t)$, $t \in \text{Prim}(A)$, is not an isomorphism. A necessary condition for this to happen is that $t$ must be a separated point in $\text{Prim}(A)$, which follows from Proposition 6.5. In case $\text{Prim}(A)$ is finite this condition is also sufficient; however, this is not always the case, see Remark 6.8 above.

### 7 A sheaf representation of $M_{\text{loc}}(A)$

For a commutative $C^*$-algebra $A$, we have

$$I(A) = M_{\text{loc}}(A) = \lim_{U \in D} C_0(U) = \text{alg}\lim_{T \in T} C_0(T),$$

where $D$ and $T$ are the filters of open dense and dense $G_\delta$ subsets of $\text{Prim}(A)$, respectively.

This is no longer true in the non-commutative setting, but we will show that both $C^*$-algebras $M_{\text{loc}}(A)$ and $I(A)$ surface as the derived algebras attached to the multiplier sheaf and the injective envelope sheaf, respectively, on the same space $\text{Prim}(A)$. The regular monotone completion of $A$, see [12], can be obtained in a similar way.

We start by associating an appropriate $C^*$-algebra to any upper semicontinuous $C^*$-bundle over a topological space $X$; see Section 5. We shall assume throughout that $X$ is a Baire space. This is of course the case for $X = \text{Prim}(A)$. Let $(A, \pi, X)$ be an usc $C^*$-bundle. We denote by $T$ the family of dense $G_\delta$ subsets of $X$, downwards directed by inclusion.
For each \( U \in \mathcal{O}_X \), define
\[
\mathfrak{D}(U) = \text{alg lim}_{T \in \mathcal{T}} \Gamma_b(T \cap U, A),
\] (7.1)
where \( \Gamma_b(T \cap U, A) \) is the \( C^* \)-algebra of bounded continuous sections on \( T \cap U \). Note that, since the norm function is merely upper semicontinuous on \( A \), we cannot guarantee that the restriction maps
\[
\Gamma_b(T \cap U, A) \to \Gamma_b(T' \cap U, A)
\]
are isomorphic for \( T' \subseteq T, T, T' \in \mathcal{T} \). Observe, in addition, that \( \mathfrak{D}(U) = \mathfrak{D}(V) \) whenever \( U = V \).

We first show that \( \mathfrak{D} \) is a presheaf of \( C^* \)-algebras over \( X \).

**Lemma 7.1** For each \( U \in \mathcal{O}_X \), the algebra \( \mathfrak{D}(U) \) is a \( C^* \)-algebra.

**Proof.** By Lemma 5.2, \( \Gamma_b(T \cap U, A) \) is a \( C^* \)-algebra for every \( T \in \mathcal{T} \). Let \( D \) be the \( C^* \)-algebra \( D = \lim_{T \in \mathcal{T}} \Gamma_b(T \cap U, A) \). Then there is a *-homomorphism \( \Phi : \mathfrak{D}(U) \to D \), and our task is to show it is an isomorphism.

We first establish the injectivity. Let \( \{s\} \in \text{alg lim} \Gamma_b(T \cap U, A) \) be an element such that \( \Phi([s]) = 0 \), where \( s \in \Gamma_b(T_0 \cap U, A) \) for a dense \( G_\delta \) subset \( T_0 \) of \( X \). For every positive integer \( n > 1 \), there is \( T_n \in \mathcal{T} \) such \( T_n \subseteq T_{n-1} \) and \( \|s|_{T \cap U} \| < 1/n \). Since \( X \) is a Baire space, \( \mathcal{T}_\infty = \bigcap_{n=1}^{\infty} T_n \) is a dense \( G_\delta \) and \( s = 0 \) on \( \mathcal{T}_\infty \cap U \). It follows that \([s] = 0\) in \( \text{alg lim} \Gamma_b(T \cap U, A) \).

Now we show that \( \Phi \) is surjective. Identifying \( \text{alg lim} \Gamma_b(T \cap U, A) \) with its image in \( D \), this amounts to show its completeness. Let \( \{s_n\}_{n \in \mathbb{N}} \) be a Cauchy sequence in \( \text{alg lim} \Gamma_b(T \cap U, A) \). By using the Baire property of \( X \), one can easily check that there is a dense \( G_\delta \) subset \( T \) of \( X \) such that \( \langle s_n|_{T \cap U} \rangle_{n \in \mathbb{N}} \) is a Cauchy sequence in the \( C^* \)-algebra \( \Gamma_b(T \cap U, A) \). The limit \( s \in \Gamma_b(T \cap U, A) \) of this Cauchy sequence satisfies that \([s] = \lim_{n \to \infty} [s_n]\), showing that the algebra \( \text{alg lim} \Gamma_b(T \cap U, A) \) is already complete. \( \square \)

**Proposition 7.2** Let \((A, \pi, X)\) be an upper semicontinuous bundle of \( C^* \)-algebras. Then \( \mathfrak{D} = \mathfrak{D}(A, \pi, X) \) is a presheaf of \( C^* \)-algebras over \( X \).

**Proof.** Clearly, for the \( C^* \)-algebras defined in (7.1), we have restriction homomorphisms \( \mathfrak{D}(U) \to \mathfrak{D}(V) \), whenever \( V \subseteq U \).

Starting with a presheaf \( \mathfrak{A} \) of \( C^* \)-algebras over a topological space \( X \), we now want to apply the above construction to the upper semicontinuous \( C^* \)-bundle \((A, \pi, X)\) associated to \( \mathfrak{A} \) via Theorem 5.6. In this way, we define the derived presheaf \( \mathfrak{D}\mathfrak{A} \) of \( \mathfrak{A} \) as the presheaf \( \mathfrak{D}(A, \pi, X) \).

It turns out that this presheaf is in fact a sheaf. To prove this, we need a known fact in Boolean algebra for which we do not have a reference.

**Lemma 7.3** Let \((p_i)_{i \in I}\) be a family of elements of a complete Boolean algebra \( \mathfrak{B} \). Then there exists a family of pairwise orthogonal elements \((q_i)_{i \in I}\) in \( \mathfrak{B} \) such that \( q_i \leq p_i \) for all \( i \) and \( \bigvee_{i \in I} q_i = \bigvee_{i \in I} p_i \).

**Proof.** We may assume that \( I = \{\alpha \mid \alpha < \gamma\} \), the set of ordinals less than a given limit ordinal \( \gamma \).

For \( \beta < \gamma \), set \( p_\beta = \bigvee_{\alpha < \beta} p_\alpha \) and \( q_\beta = p_{\beta+1} - P_\beta \). Clearly, \((q_\beta)_{\beta < \gamma}\) is a family of pairwise orthogonal elements in \( \mathfrak{B} \). Moreover,
\[
q_\beta = p_{\beta+1} - p_\beta = \bigvee_{\alpha \leq \beta} p_\alpha - \bigvee_{\alpha < \beta} p_\alpha = p_\beta \left(1 - \bigvee_{\alpha < \beta} p_\alpha\right) \leq p_\beta
\]
so that \( q_\beta \leq p_\beta \) for all \( \beta < \gamma \).

Finally we check that \( \bigvee_{i \in I} q_i = \bigvee_{i \in I} p_i \). Since \( \bigvee_{i \in I} q_i \leq \bigvee_{i \in I} p_i \), it will suffice to verify by transfinite induction that \( p_\beta \leq \bigvee_{\alpha \leq \beta} q_\alpha \) for all \( \beta < \gamma \). Assuming that \( p_\lambda \leq \bigvee_{\alpha \leq \lambda} q_\alpha \) for all \( \lambda \leq \beta \), we shall show that \( p_{\beta+1} \leq \bigvee_{\alpha \leq \beta+1} q_\alpha \). Indeed,
\[
p_{\beta+1} = p_{\beta+1} - p_\beta P_{\beta+1} + p_\beta P_{\beta+1}
= p_{\beta+1} \vee P_{\beta+1} - P_{\beta+1} + p_{\beta+1} P_{\beta+1}
\leq q_{\beta+1} + \bigvee_{\alpha \leq \beta} q_\alpha = \bigvee_{\alpha \leq \beta+1} q_\alpha.
\]
where we used that $P_{\beta+1} \leq \bigvee_{\alpha \leq \beta} q_\alpha$ by induction hypothesis. If $\beta$ is a limit ordinal then $\bigvee_{\alpha < \beta} q_\alpha = \bigvee_{\alpha < \beta} p_\alpha = P_\beta$ by induction hypothesis, and we have

$$
\bigvee_{\alpha \leq \beta} q_\alpha = q_\beta + \bigvee_{\alpha < \beta} q_\alpha = P_{\beta+1} = P_\beta + P_\beta = P_{\beta+1}
$$

which implies that $p_\beta \leq \bigvee_{\alpha < \beta} q_\alpha$. This concludes the proof. \hfill \Box

**Proposition 7.4** Let $\mathfrak{A}$ be a presheaf of unital $C^*$-algebras over a Baire space $X$. Then the derived presheaf $\mathfrak{D}_\mathfrak{A}$ is a sheaf of unital $C^*$-algebras over $X$.

**Proof.** Let $\mathcal{D} = \mathfrak{D}_\mathfrak{A}$ denote the derived presheaf of $\mathfrak{A}$. To show the gluing property, we start with some preliminary observations. For each open subset $U$ of $X$, let $p_U$ be the class in $\mathfrak{D}(X)$ of the characteristic function of $U$, seen as a continuous section on the union of $U$ and the interior of $X \setminus U$. Then $p_U$ is a central projection in $\mathfrak{D}(X)$ and $p_U \mathfrak{D}(X) = \mathfrak{D}(U)$. Observe also that $p_U = p_V$ whenever $\overline{U} = \overline{V}$, so that the set of all the projections $\{p_U\}_{U \in \mathcal{O}_X}$ is a complete Boolean algebra $\mathfrak{B}$ isomorphic to the Boolean algebra of regular open subsets of $X$.

If we are given a collection $\{U_i \mid i \in I\}$ of pairwise disjoint open subsets of $X$ and a bounded family $(a_i)_{i \in I}$ of elements in $\mathfrak{D}(X)$ with $p_{U_i} a_i = a_i$ for all $i$, where $p_i = p_{U_i}$, then there is a unique $a = \sum a_i p_i \in p \mathfrak{D}(X)$ such that $p_i a = a_i$ for every $i \in I$, where $p$ denotes the supremum of the family $(p_i)$. To show the gluing property of $\mathfrak{D}$, we consider a set of projections $(p_i)_{i \in I}$ corresponding to open sets $U_i$ of $X$ and a bounded set of elements $(a_i)_{i \in I}$ with $p_i a_i = a_i$ and $p_j p_i a_i = p_j p_i a_j$ for every $i, j \in I$. Then we have to show that there is a unique $a \in p \mathfrak{D}(X)$ such that $a p_i = a_i p_i$ for all $i \in I$, where $p$ is the supremum of the family $(p_i)$.

By Lemma 7.3, there is a family $(q_i)_{i \in I}$ of pairwise orthogonal projections in the complete Boolean algebra $\mathfrak{B}$ such that $q_i \leq p_i$ for all $i$ and $\bigvee_{i \in I} q_i = \bigvee_{i \in I} p_i$. It follows from the previous observation that there is a unique element $a = \sum a_i q_i \in p \mathfrak{D}(X)$ such that $a q_i = a_i q_i$ for all $i$, for each $i \in I$,

$$
ap_i = \left( \sum_{j \in I} a_j q_j \right) p_i = \sum_{j \in I} a_j p_i q_j = \sum_{j \in I} a_j p_i q_j = a_i p_i \left( \sum_{j \in I} q_j \right) = a_i p_i,
$$

from which we conclude the result. \hfill \Box

The process of passing to the derived sheaf obeys the following functorial property.

**Proposition 7.5** Let $X$ be a Baire space. The map $\mathfrak{D}$ defines a functor

$$
\mathfrak{D} : \mathcal{PSh}(X, C^*_+ \mathbb{C}) \rightarrow \mathcal{Sh}(X, C^*_+ \mathbb{C}).
$$

If $\iota : \mathfrak{A} \rightarrow \mathfrak{B}$ is a faithful natural transformation (that is, $\iota_U : \mathfrak{A}(U) \rightarrow \mathfrak{B}(U)$ is injective for every $U \in \mathcal{O}_X$), then $\mathfrak{D}(\iota) : \mathfrak{D}_\mathfrak{A} \rightarrow \mathfrak{D}_\mathfrak{B}$ is also faithful. For every presheaf $\mathfrak{A}$ of unital $C^*$-algebras over $X$, the sheaf $\mathfrak{D}_\mathfrak{A}$ is a $\mathfrak{D}_{\iota(X)}$-sheaf.

**Proof.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be two presheaves of unital $C^*$-algebras over $X$, and let $\mathfrak{A}$ and $\mathfrak{B}$ be the usc $C^*$-bundles associated to $\mathfrak{A}$ and $\mathfrak{B}$, respectively (Theorem 5.6). We first show that a natural transformation $F : \mathfrak{A} \rightarrow \mathfrak{B}$ yields a continuous bundle map $\tilde{F} : \mathfrak{A} \rightarrow \mathfrak{B}$. In fact, $\tilde{F}$ induces a morphism of $C^*$-algebras

$$
F_1 : A_1 \rightarrow B_1
$$

between the fibres of the $C^*$-bundles for each $t \in X$.

We show that $\tilde{F}$ is continuous. Let $a \in A$ and set $t_0 := \pi(a)$. Let $V(U, s, \varepsilon)$ be a basic neighbourhood of $F_{t_0}(a)$ in $\mathfrak{B}$, so that $U$ is an open neighbourhood of $t_0, s \in \mathfrak{B}(U)$ and $\|F_{t_0}(a) - s(t_0)\| < \varepsilon$.

Set $\varepsilon' := \|F_{t_0}(a) - s(t_0)\|$ and choose $\delta > 0$ such that $\varepsilon' + 2\delta < \varepsilon$. There are an open neighbourhood $W_1$ of $t_0$, with $W_1 \subseteq U$, and $s_1 \in \mathfrak{A}(W_1)$ such that

$$
\|s_1(t_0) - a\| < \delta.
$$

It follows that

$$
\|F_{W_1}(s_1)(t_0) - F_{t_0}(a)\| = \|F_{t_0}(s_1(t_0) - a)\| \leq \|s_1(t_0) - a\| < \delta.
$$

(7.3)
Therefore
\[ \|(s_{|W_2} - F_{W_2}(s_1))(t_0)\| \leq \|s(t_0) - F_{\alpha_0}(a)\| + \|F_{W_2}(s_1)(t_0) - F_{\alpha_0}(a)\| < \varepsilon' + \delta. \]
Consequently, there is an open subset \( W_2 \) of \( X \) with \( t_0 \in W_2 \subseteq W_1 \) such that
\[ \|s_{|W_2} - F_{W_2}(s_{|W_2})\| < \varepsilon' + \delta. \]
(7.4)
In order to show that \( \tilde{\mathcal{F}} \left(V(W_2, s_1|W_2, \delta)\right) \subseteq V(U, s, \varepsilon) \) take \( a' \in V(W_2, s_1|W_2, \delta) \). We have \( \|a' - s_1(\pi(a'))\| < \delta \) and so, using (7.4), we obtain
\[
\|\tilde{\mathcal{F}}(a') - s(\pi(a'))\| \leq \|F_{\pi(a')}(a') - F_{W_2}(s_1|W_2)(\pi(a'))\| + \|F_{W_2}(s_1|W_2)(\pi(a')) - s(\pi(a'))\|
\leq \|a' - s_1(\pi(a'))\| + \|s_{|W_2} - F_{W_2}(s_{|W_2})\| < \varepsilon' + 2\delta < \varepsilon,
\]
which shows that \( \tilde{\mathcal{F}}(a') \in V(U, s, \varepsilon) \), as claimed.

We now define \( \mathcal{D}(F) : \mathcal{D}_\Theta \rightarrow \mathcal{D}_\Theta \) by
\[
\mathcal{D}(F)(s) = [\tilde{\mathcal{F}} \circ s] \text{ for } [s] \in \mathcal{D}_\Theta(U), \text{ } U \in \mathcal{O}_X.
\]
It is easy to show that \( \mathcal{D}(\iota) \) is faithful whenever \( \iota \) is a faithful natural transformation. Since the C*-algebra \( \Gamma_b(T \cap U, A) \) is a Banach module over \( G_b(T \cap U) \) for every \( T \in \mathcal{O}_X \) and every \( U \in \mathcal{O}_X \), we see that \( \mathcal{D}_\Theta \) is automatically a \( \mathcal{D}_\Theta(X) \)-sheaf.

We are ready to prove our main result which states that the derived sheaf of the multiplier sheaf of a C*-algebra \( A \) agrees with the sheaf associating to each \( U \in \mathcal{O}_{\text{Prim}(A)} \) the algebra \( M_{\text{loc}}(A(U)) = p_U : M_{\text{loc}}(A) \). We shall call the latter sheaf the local multiplier sheaf of \( A \) and denote it by \( \mathcal{M}_{\text{loc}}A \).

**Theorem 7.6** There is a natural isomorphism between the derived sheaf of the multiplier sheaf of a C*-algebra \( A \) and the local multiplier sheaf of \( A \), that is, \( \mathcal{D}_{\text{M}_{\text{loc}}A} \cong \mathcal{M}_{\text{loc}}A \).

**Proof.** Since, for \( U \in \mathcal{O}_{\text{Prim}(A)} \), we have \( \mathcal{M}_{\text{loc}}A(U) = M_{\text{loc}}(A(U)) \), and the dense \( G_\delta \)'s of \( U \) are the intersections of dense \( G_\delta \)'s of \( \text{Prim}(A) \) with \( U \), it suffices to show that there is an isomorphism \( M_{\text{loc}}(A) \cong \text{alg lim}_{T \in \mathcal{T}} G_\delta(T, A) \), where \( T \) is the family of dense \( G_\delta \)'s of \( \text{Prim}(A) \) and \( (A, \pi, \text{Prim}(A)) \) is the C*-bundle associated to \( \mathcal{M}_A \).

Let \( U \) be a dense open subset of \( \text{Prim}(A) \). There is a natural injective *-homomorphism \( \mu_U : M(A(U)) \rightarrow \Gamma_b(U, A) \), see Section 5, (5.2). We claim that the composition
\[
M(A(U)) \rightarrow \Gamma_b(U, A) \rightarrow \text{alg lim}_{T \in \mathcal{T}} \Gamma_b(T, A)
\]
is an isometry, that is, given \( x \in M(A(U)) \), \( \|x\| = \|\mu_U(x)\| \) for every dense \( G_\delta \) subset \( T \) of \( U \). Observe that \( T \) is a dense subset of \( \text{Prim}(M(A(U))) \), because \( U \) is dense in \( \text{Prim}(M(A(U))) \). Therefore, \( \|x\| = \sup_{t \in T} \|x + \tilde{t}\| \), where \( \tilde{t} \in \text{Prim}(M(A(U))) \) is the primitive ideal of \( M(A(U)) \) corresponding to \( t \). Recall that we have a *-homomorphism
\[
\varphi_t : A_t \rightarrow M_{\text{loc}}(A/t),
\]
and that, by the commutative square (6.2), we have \( x + \tilde{t} = \varphi_t(\mu_U(x)(t)) \) for every \( t \in U \). This entails that
\[
\|x\| = \|\mu_U(x)\| \geq \|\mu_U(x)(t)\| \geq \sup_{t \in T} \|\varphi_t(\mu_U(x)(t))\| = \sup_{t \in T} \|x + \tilde{t}\| = \|x\|
\]
and hence the desired isometry.

Since the maps \( M(A(U)) \rightarrow \text{alg lim}_{T \in \mathcal{T}} \Gamma_b(T, A) \) are obviously compatible with the restriction maps \( M(A(U)) \rightarrow M(A(V)) \) for \( V \subseteq U \) dense open subsets of \( \text{Prim}(A) \), we get an injective *-homomorphism \( M_{\text{loc}}(A) \rightarrow \text{alg lim}_{T \in \mathcal{T}} \Gamma_b(T, A) \). We have to show that this map is surjective and for this, it suffices to check
that its image is dense. Given \( \varepsilon > 0 \) and \( s \in \Gamma_\mathcal{A}(T, \mathcal{A}) \), with \( T \in \mathcal{T} \), take a maximal family \( \{ U_i \}_{i \in I} \) of pairwise disjoint open subsets of \( \text{Prim}(\mathcal{A}) \) such that there are elements \( a_i \in M(\mathcal{A}(U_i)) \) with the property that \( ||s_{T \cap U_i} - \mu_{U_i}(a_i)_{T \cap U_i}|| < \varepsilon \). If \( U = \bigcup_{i \in I} U_i \) is not dense in \( \text{Prim}(\mathcal{A}) \), there is a non-empty open subset \( W \) of \( \text{Prim}(\mathcal{A}) \) disjoint from \( U \), and, since \( T \) is dense, there is \( t_0 \in T \cap W \). Let \( W_1 \) be an open neighbourhood of \( t_0 \) with \( W_1 \subseteq W \) and \( a \in M(\mathcal{A}(W_1)) \) such that \( ||s(t_0) - \mu_{W_1}(a)(t_0)|| < \varepsilon \). Using the continuity of \( s \) and of \( \mu_{W_1}(a) \), we find a smaller neighbourhood \( W_2 \) of \( t_0 \) such that

\[
||s_{T \cap W_2} - \mu_{W_2}(a)_{T \cap W_2}|| < \varepsilon.
\]

This contradicts the maximality of the family \( \{ U_i \}_{i \in I} \).

It follows that \( U \) is dense in \( \text{Prim}(\mathcal{A}) \), and the element \( a = \sum a_i \in M(\mathcal{A}(U)) \) (which exists by [1, Lemma 3.3.6]) satisfies \( ||s_{T \cap U} - \mu_{U}(a)_{T \cap U}|| < \varepsilon \), as desired.

The following analogue of the above theorem holds for the injective envelope sheaf.

**Theorem 7.7** For every \( C^*\)-algebra \( \mathcal{A} \), we have \( \mathcal{D}_\mathcal{A} \cong \mathcal{J}_\mathcal{A} \) as sheaves over \( \text{Prim}(\mathcal{A}) \).

**Proof.** Let \( U \in \mathcal{C}_{\text{Prim}(\mathcal{A})} \). Since \( \mathfrak{M}_\mathcal{A}(U) = M(\mathcal{A}(U)) \to I(\mathcal{A}(U)) = p_U I(\mathcal{A}) = \mathcal{J}_\mathcal{A}(U) \) defines a faithful natural transformation \( \mathfrak{M}_\mathcal{A} \to \mathcal{J}_\mathcal{A} \), Proposition 7.5 yields a faithful natural transformation \( \mathfrak{M}_\mathcal{J}_\mathcal{A} \to \mathcal{D}_\mathcal{J}_\mathcal{A} \). This provides us with the following commutative diagram

\[
\begin{array}{ccc}
M(\mathcal{A}(U)) & \longrightarrow & \mathfrak{M}_\mathcal{J}_\mathcal{A}(U) \\
\downarrow & & \downarrow \\
I(\mathcal{A}(U)) \end{array}
\]

in which the composition of the horizontal maps is injective, by the proof of Theorem 7.6. To show the injectivity of \( I(\mathcal{A}(U)) \to \mathcal{D}_\mathcal{J}_\mathcal{A}(U) \), it therefore suffices that every non-zero closed ideal in \( I(\mathcal{A}(U)) = I(M(\mathcal{A}(U))) \) must have non-zero intersection with \( M(\mathcal{A}(U)) \), which is, e.g., [3, Proposition 2.12]. As a result, we can consider \( \mathcal{J}_\mathcal{A}(U) \) as unital \( C^*\)-subalgebra of \( \mathcal{D}_\mathcal{J}_\mathcal{A}(U) \). The proof that \( \mathcal{J}_\mathcal{A}(U) \) is dense in \( \mathcal{D}_\mathcal{J}_\mathcal{A}(U) \) (and thus equal to \( \mathcal{D}_\mathcal{J}_\mathcal{A}(U) \)) follows verbatim the argument of the corresponding statement in the proof of Theorem 7.6 and is hence omitted.

It was observed in [10], see also [3, Section 4], that \( M_{\text{loc}}(\mathcal{A}) \) canonically embeds into \( I(\mathcal{A}) \). Here comes the sheaf-theoretic analogue of this result.

**Corollary 7.8** Let \( \mathcal{A} \) be a \( C^*\)-algebra. Then the local multiplier sheaf of \( \mathcal{A} \) canonically embeds into the injective envelope sheaf, that is, \( \mathfrak{M}_{\text{loc}} \mathcal{A} \hookrightarrow \mathcal{J}_\mathcal{A} \).

**Proof.** It is noted in the above proof that we have a faithful natural transformation \( \mathfrak{M}_{\mathfrak{M}_\mathcal{A}} \to \mathcal{D}_\mathcal{J}_\mathcal{A} \). Combining this with Theorems 7.6 and 7.7 thus yields the result.

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