WHEN ARE ALL BOUNDED OPERATORS BETWEEN CLASSICAL BANACH LATTICES REGULAR?

A.W. WICKSTEAD

ABSTRACT. We give a short survey of when all bounded operators between two Banach lattices are the difference of two positive operators, in the case that both Banach lattices are some of the classical Banach lattices.

We refer the reader to [8] for the basic theory and terminology about Banach lattices. *Regular* operators are those that can be written as the difference of two positive operators. We write $\mathcal{L}(X,Y)$ for the space of bounded operators from X into Y and $\mathcal{L}^r(X,Y)$ for the corresponding space of regular operators. In 1936 Kantorovich, [5], proved that every regular operator must be bounded and made a start on asking when the converse is true. I.e. when is every bounded linear operator from X into Y regular? There is no simple answer to this question. In order to illustrate our current state of knowledge, I will tell you what happens when X and Y are selected from the classical Banach lattices ℓ_1, $L_1([0,1])$, ℓ_p $(1 < p < \infty)$, $L_p([0,1])$ $(1 < p < \infty)$, c_0, c, ℓ_∞ or $C([0,1])$. In the course of doing this we will meet (at least versions of) many of the known results in this field. The bare facts are presented in the following table. Each cell in this array contains not only an indication of whether there is equality or not but also a number that is a reference to the justification of that statement in this note.

X \ Y	$L_q([0,1])$ or ℓ_q $(1 \le q < \infty)$	ℓ_∞	c_0	c	$C([0,1])$
ℓ_1	$= (2)$	$= (1)$	$= (3)$	$= (3)$	$= (3)$
$L_1([0,1])$	$= (2)$	$= (1)$	$\ne (13)$	$= (5)$	$= (5)$
ℓ_p $(1 < p < \infty)$	$\ne (6)$	$= (1)$	$= (4)$	$= (4)$	$= (4)$
$L_p([0,1])$ $(1 < p < \infty)$	$\ne (6)$	$= (1)$	$\ne (13)$	$\ne (14)$	$\ne (15)$
c_0	$\ne (6)$	$= (1)$	$= (4)$	$= (4)$	$= (4)$
c	$\ne (6)$	$= (1)$	$\ne (8)$	$= (10)$	$= (10)$
ℓ_∞	$\ne (6)$	$= (1)$	$\ne (8)$	$\ne (10)$	$\ne (10)$
$C([0,1])$	$\ne (6)$	$= (1)$	$\ne (8)$	$\ne (10)$	$\ne (10)$

All that remains is to justify all of these claims. We will start with the cases where there is equality. The earliest positive result is due to Kantorovich in 1936, [5], where he proved:

Theorem 1. *If Y is a Dedekind complete Banach lattice with a strong order unit then $\mathcal{L}(X,Y) = \mathcal{L}^r(X,Y)$ for all Banach lattices X.*

Amongst our examples only ℓ_∞ has both of these properties. There are other Banach lattices Y for which the conclusion is true, see [2], but no other Dedekind

---

complete Banach lattices, [4]. Another case that almost belongs to the pre-history of this subject is the following:

Theorem 2. *If X is an AL-space and Y has a Levi norm then $\mathcal{L}(X,Y) = \mathcal{L}^r(X,Y)$.*

Kantorovich and Vulikh proved this with a slightly stronger assumption on Y in [6]. Synnatzschke formulated the current version in [11]. Again, it is shown in [4] that amongst Dedekind complete Banach lattices Y this is the best possible result. The spaces ℓ_p and $L_p([0,1])$ have a Levi norm for any $p \in [1,\infty]$. The folk-lore of this topic also contains the result:

Theorem 3. *If X is an atomic AL-space then, for all Banach lattices Y, $\mathcal{L}(X,Y) = \mathcal{L}^r(X,Y)$.*

Apart from changing the norm on X to an equivalent one, this is the best possible result, see van Rooij [10]. In a similar vein, in [12], Xiong proved:

Theorem 4. *If X is atomic with an order continuous norm and Y is an AM-space then $\mathcal{L}(X,Y) = \mathcal{L}^r(X,Y)$.*

The class of AM-spaces can only be widened by allowing an equivalent norm. I suspect that the condition of being atomic with an order continuous norm cannot be weakened, but that remains an open problem. Our final positive result, which is rather elementary, is:

Theorem 5. *If X is an AL-space and Y has a strong order unit then $\mathcal{L}(X,Y)$ has a strong order unit and therefore $\mathcal{L}(X,Y) = \mathcal{L}^r(X,Y)$.*

You may have noticed that, so far, all the results that guarantee that $\mathcal{L}(X,Y) = \mathcal{L}^r(X,Y)$ involve either X being (up to an equivalent norm) an AL-space and/or Y being an AM-space. For some time it was thought that one of these conditions was necessary. In [1], Abramovich gave an example to show that this was not true. Nevertheless, there are results in that kind of vein. I won't give a precise definition of a Banach lattice E being *finitely lattice representable* in F, but simply point out that it certainly holds if E is isometric to a sublattice of F.

Theorem 6. *If ℓ_p is finitely lattice representable in Y for $p \in [1,\infty)$ or ℓ_∞ is not finitely lattice representable in Y and $\mathcal{L}(X,Y) = \mathcal{L}^r(X,Y)$ then X is (up to an equivalent norm) an AL-space.*

The first version was proved by Abramovich and Janovsky in [3], and the second by Ørno in [9]. A result of Krivine, [7], says that given any Banach lattice E, there is $p \in [1,\infty]$ such that ℓ_p is finitely lattice representable in E, so the two versions of Theorem 6 are actually equivalent.

Similarly we have:

Theorem 7. *If ℓ_p is finitely lattice representable in X for $p \in (1,\infty]$ or ℓ_1 is not finitely lattice representable in X and $\mathcal{L}(X,Y) = \mathcal{L}^r(X,Y)$ then Y is (up to an equivalent norm) an AM-space.*

Again, the first version is due to Abramovich and Janovsky. In [12] Xiong investigated the subject matter of this talk and paid special attention to the case that $X = C(K)$ and $Y = C(\Omega)$ or $Y = c_0$.

Theorem 8. *If K is an infinite compact Hausdorff space then $\mathcal{L}\big(C(K),c_0\big) \neq \mathcal{L}^r\big(C(K),c_0\big)$.*

Definition 9. A compact Hausdorff space K is an $X(n)$ space if it is the disjoint union of n open and closed subsets each of which is the one-point compactification of a discrete space. The interval $[0,1]$ is not an $X(n)$ space for any $n \in \mathbb{N}$, nor is the Stone-Čech-compactification of $\mathbb{N}$.

Theorem 10 (Xiong)**.** *If K is a compact Hausdorff space then the following are equivalent:*

(1) *For some $p \in \mathbb{N}$, K is an $X(p)$-space.*
(2) *For every compact Hausdorff space Ω, $\mathcal{L}\big(C(K), C(\Omega)\big) = \mathcal{L}^r\big(C(K), C(\Omega)\big)$.*
(3) *$\mathcal{L}\big(C(K), C([0,1])\big) = \mathcal{L}^r\big(C(K), C([0,1])\big)$.*
(4) *$\mathcal{L}\big(C(K), c\big) = \mathcal{L}\big(C(K), c\big)$.*

The implication (3) implies (4) follows from a special case of a general result of Xiong, [12], Corollary 1.3, that we will need again later:

Proposition 11. *If X is any Banach lattice and $\mathcal{L}\big(X, C([0,1])\big) = \mathcal{L}^r\big(X, C([0,1])\big)$ then $\mathcal{L}(X, c) = \mathcal{L}^r(X, c)$.*

The time has come when we must start looking at some actual examples of bounded operators that are not order bounded, to take care of the remaining cases.

Definition 12. The *Rademacher* functions on $[0,1]$ are defined by $r_n(t) = \mathrm{sgn}\big(\sin(2^n \pi t)\big)$ where sgn denotes the sign function.

The Rademacher functions lie in $L_p([0,1])$ for all p (including $p = \infty$). They form an orthonormal sequence in the Hilbert space $L_2([0,1])$. Furthermore for finite p the functionals that they define on $L_p([0,1])$ by integration converge weak* to 0. I.e. for all $f \in L_p([0,1])$, $\phi_n(f) = \int_0^1 r_n(t)f(t)\, dt \to 0$ as $n \to \infty$, as long as $p \in [1, \infty)$. You can also see either by direct calculation or from the uniform boundedness principle that (ϕ_n) is a bounded sequence in $L_p([0,1])^*$.

Proposition 13. *If $p \in [1, \infty)$ there is a bounded linear operator $T : L_p([0,1]) \to c_0$ which is not regular.*

Proof. Define an operator $T : L_p([0,1]) \to c_0$ by $Tx = \big(\phi_n(x)\big)$. T is certainly bounded. Note that for $n \geq 1$ $T(r_n) = \boldsymbol{e}_n$, a basis vector in c_0, because of the orthonormality of the Rademacher functions, and that $T(r_0) = 0$ so that $T(r_0 + r_n) = \boldsymbol{e}_n$. As $0 \leq r_0 + r_n \leq 2r_0$, if we had $U \geq T, 0$ then

$$U(2r_0) \geq U(r_0 + r_n) \geq T(r_0 + r_n) = \boldsymbol{e}_n$$

for all $n \in \mathbb{N}$, which is inconsistent with $U(2r_0)$ lying in c_0. It follows that T is not regular after all. $\square$

This kind of example won't deal with the case of operators from $L_p([0,1])$ into c, though. Indeed, we already know that if $p = 1$ then all bounded operators are regular. An example in the case $p > 1$ is slightly more difficult.

Proposition 14. *If $1 < p < \infty$ then $\mathcal{L}^r\big(L_p([0,1]), c\big) \neq \mathcal{L}^r\big(L_p([0,1]), c\big)$.*

Proof. A bounded linear operator T from any Banach space X into c may be described by a bounded weak* convergent sequence of functionals (f_n) in X^*, with limit $f \in X^*$ *via* the formula $Tx = \big(f_n(x)\big)$.

Let $p^{-1} + q^{-1} = 1$, so that $1 < q < \infty$. If $n \in \mathbb{N}$, write $n = 2^j k$ where k is odd and then define

$$f_n(t) = \begin{cases} 2^{j/q} j^{-1/q} r_k(2^{-j}t) & t \in [0, 2^{-j}] \\ 0 & t \in (2^{-j}, 1] \end{cases}$$

so that $|f_n| = 2^{j/q} j^{-1/q} \chi_{[0, 2^{-j}]}$ and hence

$$\|f_n\|_q = (2^j j^{-1} 2^{-j})^{1/q} = j^{-1/q}.$$

The sequence (f_n) is certainly bounded, so we may define $T : L_p([0,1]) \to \ell_\infty$ by the formula $Tx = \big(\int_0^1 f_n(t)x(t)\, dt\big)$. In fact, $Tx \in c_0 \subseteq c$ for all $x \in L_p([0,1])$. To

see this, pick any x with $\|x\| \leq 1$ and $\epsilon > 0$. First choose $j_0 \in \mathbb{N}$ such that $j > j_0$ implies that $j^{-1/q} < \epsilon$, so that (if k is odd) $|(Tx)_{2^j k}| \leq \|f_{2^j k}\|_q = j^{-1/q} < \epsilon$. For each $j \leq j_0$ there is k_j such that $k > k_j$ implies that $|(Tx)_{2^j k}| = |f_{2^j k}(x)| < \epsilon$. We now see that $|(Tx)_{2^j k}| < \epsilon$ except for the finite number of integers $2^j k$ with $j \leq j_0$ and $k \leq k_j$.

If T were regular, there would be $U : L_p([0,1]) \to c$ with $U \geq \pm T$. Write $g_n(x) = (Ux)_n$ and $g(x) = \lim_{n \to \infty} g_n(x)$. As $U \geq \pm T$, $g_n \geq |f_n|$ for all $n \in \mathbb{N}$. If we fix j then we see that

$$|g_{2^j k}| \geq |f_{2^j k}| = 2^{j/q} j^{-1/q} \chi[0, 2^{-j}]$$

for all odd integers k. Letting $k \to \infty$ through the odd integers, and using the fact that the usual positive cone in $L_q([0,1])$ is weak*-closed, we see that $g \geq 2^{j/q} j^{-1/q} \chi[0, 2^{-j}]$. This holds for all $j \in \mathbb{N}$, so that

$$\|g\|_q^q \geq \sum_{j=1}^{\infty} 2^{-j-1} (2^{j/q} j^{-1/q})^q = \sum_{j=1}^{\infty} \frac{1}{2j} = \infty,$$

so that $g \notin L_q([0,1])$, U does not exist and T is not regular. $\square$

Recalling Proposition 11 we also see that

Proposition 15. *If* $1 < p < \infty$ *then* $\mathcal{L}^r\big(L_p([0,1]), C([0,1])\big) \neq \mathcal{L}^r\big(L_p([0,1]), C([0,1])\big).$

REFERENCES

[1] Ju. A. Abramovič, *The space of operators that act between Banach lattices*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **73** (1977), 188–192, 234 (1978) (Russian, with English summary). Investigations on linear operators and the theory of functions, VIII. MR513175 (80a:47069)

[2] Ju. A. Abramovič and V. A. Gejler, *On a question of Fremlin concerning order bounded and regular operators*, Colloq. Math. **46** (1982), no. 1, 15–17. MR672357 (84b:46008)

[3] Ju. A. Abramovič and L. P. Janovskiĭ, *Application of Rademacher systems to operator characterizations of Banach lattices*, Colloq. Math. **46** (1982), no. 1, 73–78. MR672365 (84h:46029)

[4] Y. A. Abramovich and A. W. Wickstead, *When each continuous operator is regular. II*, Indag. Math. (N.S.) **8** (1997), no. 3, 281–294. MR1622216 (99c:47046)

[5] L.V. Kantorovich, *Concerning the general theory of operations in partially ordered spaces*, Dok. Akad. Nauk. SSSR **1** (1936), 271–274 (Russian).

[6] L. Kantorovitch and B. Vulich, *Sur la représentation des opérations linéaires*, Compositio Math. **5** (1938), 119–165 (French). MR1556991

[7] J. L. Krivine, *Sous-espaces de dimension finie des espaces de Banach réticulés*, Ann. of Math. (2) **104** (1976), no. 1, 1–29. MR0407568 (53 #11341)

[8] Peter Meyer-Nieberg, *Banach Lattices*, Universitext, Springer-Verlag, Berlin, 1991. MR1128093 (93f:46025)

[9] Peter Ørno, *On Banach lattices of operators*, Israel J. Math. **19** (1974), 264–265. MR0374859 (51 #11055)

[10] A. C. M. van Rooij, *On the space of all regular operators between two Riesz spaces*, Nederl. Akad. Wetensch. Indag. Math. **47** (1985), no. 1, 95–98. MR783009 (86k:46011)

[11] Ju. Synnačke, *The operator that is conjugate to a regular one, and certain applications of it to the question of the complete continuity and weak complete continuity of regular operators*, Vestnik Leningrad. Univ. **1** (1972), 60–69 (Russian, with English summary). MR0312314 (47 #876)

[12] Hong Yun Xiong, *On whether or not $\mathcal{L}(E,F) = \mathcal{L}^r(E,F)$ for some classical Banach lattices E and F*, Nederl. Akad. Wetensch. Indag. Math. **46** (1984), no. 3, 267–282. MR763464 (86a:47042)

PURE MATHEMATICS RESEARCH CENTRE, QUEENS UNIVERSITY BELFAST, NORTHERN IRELAND.
E-mail address: `A.Wickstead@qub.ac.uk`