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Spectral isometries into commutative Banach algebras

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Dedicated to the memory of James E. Jamison.

Abstract. We determine the structure of spectral isometries between unital Banach algebras under the hypothesis that the codomain is commutative.

1. Introduction

Spectral isometries, that is, spectral radius-preserving linear mappings, are the non-selfadjoint analogues of isometries between unital C^*-algebras. Every Jordan isomorphism preserves the spectrum of each element (of the domain), hence the spectral radius. Under the assumption that it is selfadjoint (that is, maps selfadjoint elements onto selfadjoint elements), it is an isometry. Kadison, in 1951, proved the converse and established a non-commutative generalization of the classical Banach–Stone theorem: Every unital surjective isometry between unital C^*-algebras is a Jordan *-isomorphism [**5**]; thus, a self-adjoint spectral isometry. Conversely, every unital surjective spectral isometry which is selfadjoint must be an isometry, an easy consequence of the Russo–Dye theorem. This, amongst others, led to the conjecture that every unital surjective spectral isometry between unital C^*-algebras is a Jordan isomorphism, see [**9**], and for a more in-depth discussion of this interplay, [**7**].

As it stands, the above conjecture is still open though there has been substantial progress towards it. It has been observed, see in particular [**10**], that the behaviour on commutative subalgebras is vital for the conjecture to hold. Moreover, under additional hypotheses, the conjecture has even been verified for certain Banach algebras; see, e.g., [**3**] and [**1**]. This motivated us to re-visit the situation for commutative Banach algebras and to fill in some loose ends in the literature. It has been known for some time that a unital surjective spectral isometry between commutative unital semisimple Banach algebras is an algebra isomorphism; this

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is Nagasawa's theorem, see, e.g., [**2**, Theorem 4.1.17]. What about, however, non-unital or non-surjective spectral isometries in this setting? The present note intends to answer these questions by a unified method.

2. Non-unital and non-surjective spectral isometries

Throughout this paper, A and B will denote unital complex Banach algebras, and we shall generally be following the notation in [**6**]. The (Jacobson) radical of A is $\mathrm{rad}(A)$ and $Z(A)$ stands for the centre of A.

Let $T\colon A \to B$ be a *spectral isometry*, that is, a linear mapping satisfying $r(Tx) = r(x)$ for all $x \in A$, where $r(\cdot)$ denotes the spectral radius. It is well known that, if T is surjective, $T\mathrm{rad}(A) = \mathrm{rad}(B)$; see [**9**, Proposition 2.11] or [**10**, Lemma 2.1]. Therefore, by passing to the quotient Banach algebras $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$, we obtain a canonically induced spectral isometry between semisimple Banach algebras. If T is not assumed to be surjective but B is commutative then, since $\mathrm{rad}(B)$ coincides with the set of all quasi-nilpotent elements in B, we still have $T\mathrm{rad}(A) \subseteq \mathrm{rad}(B)$ and the same argument applies. As a result, we shall henceforth assume that our Banach algebras are semisimple (instead of formulating the results "modulo the radical").

Suppose T is a surjective spectral isometry. Then $TZ(A) = Z(B)$ [**9**, Proposition 4.3], a fact that turned out to be very useful in the non-commutative setting. If T is not surjective, once again the assumption that B is commutative will prove itself to be expedient.

Our approach exploits the close relationship between spectral isometries on semisimple commutative Banach algebras and isometries on Banach function algebras; on the latter, there is a vast literature, see, e.g., [**4**]. The main tool will be a version of Novinger's theorem and a consequence of it which was originally obtained by deLeeuw, Rudin and Wermer. For convenience, we will formulate this in one result. Recall first that the *Choquet boundary* $\mathrm{ch}(E)$ of a linear space E of continuous functions on a compact Hausdorff space X is defined as

$$\mathrm{ch}(E) = \{t \in X \mid \epsilon_t \text{ is an extreme point of } E_1^*\},$$

where E_1^* denotes the dual unit ball and ϵ_t is the point evaluation at t.

Theorem 2.1 ([**4**], Theorem 2.3.10 and Corollary 2.3.16). *Let X and Y be compact Hausdorff spaces and denote by $C(X)$ and $C(Y)$ the Banach algebras of continuous complex-valued functions on X and Y, respectively. Let $E \subseteq C(X)$ be a subspace which separates the points of X and contains the constant functions. Suppose T is a linear isometry from E onto a subspace $F \subseteq C(Y)$. Then there exist a function $h \in C(Y)$, which is unimodular on $\mathrm{ch}(F)$, and a continuous function φ from $\mathrm{ch}(F)$ onto $\mathrm{ch}(E)$ such that*

$$Tf(t) = h(t)\, f(\varphi(t)) \quad \text{for all } f \in E \quad \text{and} \quad t \in \mathrm{ch}(F). \tag{2.1}$$

If, moreover, E and F are unital subalgebras then h is unimodular on Y and T_1 defined by $T_1 f = \overline{h}\, Tf$, $f \in E$ is an algebra isomorphism from E onto F.

In particular, if the isometry T is *unital*, that is, $T1 = 1$, T will be an algebra isomorphism from E onto F if and only if F is a subalgebra of $C(Y)$. In general, the image of a unital isometry defined on a subalgebra of $C(X)$ need not be a subalgebra of $C(Y)$. Since this fact partly motivates our paper, we recall one of the well-known examples.

Example 2.2 (McDonald, see [**4**], Example 2.3.17). Let φ_1, φ_2 be continuous functions from the compact Hausdorff space Y into the compact Hausdorff space X. Define $T\colon C(X) \to C(Y)$ by $Tf(t) = \frac{1}{2}\big(f(\varphi_1(t)) + f(\varphi_2(t))\big)$, $t \in Y$. Let $\Gamma = \{t \in Y \mid \varphi_1(t) = \varphi_2(t)\}$. If $\varphi_1(\Gamma) = X$ then T is a unital isometry. However, $F = \operatorname{im} T$ is not a subalgebra of $C(Y)$ in general since $\operatorname{ch}(F) = \Gamma$ which may be smaller than Y. Indeed, for $t \in Y \setminus \Gamma$ take $f \in C(X)$ such that $f(\varphi_1(t)) = 1$ and $f(\varphi_2(t)) = 0$. Then

$$\big(Tf\,T(1-f)\big)(t) = -\frac{1}{4} \quad \text{whereas} \quad T\big(f(1-f)\big)(t) = 0.$$

The Choquet boundary of a subspace $F \subseteq C(Y)$ is always a boundary for F in the sense that, for each $g \in F$, there is $t \in \operatorname{ch}(F)$ such that $\|g\| = |g(t)|$ (Phelps' theorem, see, e.g., [**4**, Theorem 2.3.8]). The above example illustrates nicely the fact that the image F of an isometry will only be an algebra if $\operatorname{ch}(F)$ is a boundary for the algebra generated by F, which is also the core of the argument to deduce the second part of Theorem 2.1 from the first.

The connection between spectral isometries on commutative Banach algebras and isometries on function algebras is of course made via Gelfand theory, but this seems not to have been exploited so far. For a unital commutative semisimple Banach algebra A we let $\Delta(A)$ denote its structure space, that is, the space of multiplicative linear functionals on A endowed with the weak* topology, also called the maximal ideal space of A. See [**6**, Chapter 2]. Recall that $\Delta(A)$ is a compact Hausdorff space.

We shall use $\Gamma_A\colon A \to C(\Delta(A))$ to denote the Gelfand transformation of A and abbreviate the image of $a \in A$ under Γ_A by $\hat{a} = \Gamma_A a$. As there is no danger of confusion, instead of $\Gamma_A A$ we will write ΓA, which is a unital (not necessarily closed) subalgebra of $C(\Delta(A))$ separating the points of $\Delta(A)$. Recall too that $r(a) = r(\hat{a}) = \|\hat{a}\|$ for all $a \in A$, and it is this fact that allows us to move from spectral isometries to isometries.

Let $T\colon A \to B$ be a spectral isometry between the unital commutative semisimple Banach algebras A and B. We define $\widehat{T}\colon \Gamma A \to \Gamma B$ by $\widehat{T} = \Gamma_B \circ T \circ {\Gamma_A}^{-1}$. Then $\widehat{T}$ is a spectral isometry which is unital, or surjective, when T has these properties. Moreover, since spectral radius and norm coincide for continuous functions, $\widehat{T}$ is in fact an isometry. Resulting from this observation, we can apply knowledge on isometries to gain information on spectral isometries, and our first application will be the following proposition.

Proposition 2.3. *Let $T\colon A \to B$ be a surjective spectral isometry between the unital Banach algebras A and B, and let $u = T1$. Then u has its spectrum in the unit circle $\mathbb{T}$.*

Proof. As $T\operatorname{rad}(A) = \operatorname{rad}(B)$ and the spectrum does not change when passing to the quotient by the radical, we may assume that both A and B are semisimple. Let $A_0 = Z(A)$ and $B_0 = Z(B)$ which are both semisimple. Let $T_0\colon A_0 \to B_0$ denote the restriction of T to A_0 which is a surjective spectral isometry [**9**, Proposition 4.3]. Applying the above transformation to the spectral isometry T_0 in this case, we obtain a surjective isometry $\widehat{T_0}\colon \Gamma A_0 \to \Gamma B_0$. The function $h \in C(\Delta(B_0))$ in Theorem 2.1, Equation (2.1) is nothing but $\widehat{T_0}1$ and has spectrum contained in $\mathbb{T}$. As $u = T1 = {\Gamma_{B_0}}^{-1}\,\widehat{T_0}\,\Gamma_{A_0}1$ it follows that $\sigma_B(u) = \sigma_{B_0}(u) \subseteq \mathbb{T}$ as claimed. $\square$

Simple examples show that the statement in the above proposition can fail for non-surjective spectral isometries even when the codomain is commutative.

As a consequence of this result, when studying surjective spectral isometries, one can always reduce to the unital case. It is customary to call an element u in a Banach algebra a *unitary* provided its spectrum $\sigma(u)$ lies in $\mathbb{T}$. (This is because such u is invertible and $\sigma(u^{-1}) \subseteq \mathbb{T}$ so u resembles a unitary operator on Hilbert space.)

Corollary 2.4. *Let $T\colon A \to B$ be a surjective spectral isometry between the unital semisimple Banach algebras A and B. Then there are a unitary $u \in Z(B)$ and a unital surjective spectral isometry $T_1\colon A \to B$ such that*

$$Ta = u\,T_1 a \qquad (a \in A). \tag{2.2}$$

Proof. Put $u = T1$ which, by Proposition 2.3, is unitary and set $T_1 a = u^{-1}\,Ta$, $a \in A$. Since u is central, for each $a \in A$,

$$r(T_1 a) \le r(u^{-1})\,r(Ta) = r(Ta) = r(a) = r(uu^{-1}\,Ta) \le r(u)\,r(u^{-1}\,Ta) = r(T_1 a)$$

whence T_1 is a unital surjective spectral isometry. $\square$

We also obtain a non-unital version of Nagasawa's theorem; see [**2**, Theorem 4.1.17].

Corollary 2.5. *Let $T\colon A \to B$ be a surjective spectral isometry between the unital commutative semisimple Banach algebras A and B. Then there are a unitary $u \in B$ and an algebra isomorphism $T_1\colon A \to B$ such that*

$$Ta = u\,T_1 a \qquad (a \in A). \tag{2.3}$$

Proof. The unital surjective spectral isometry $T_1\colon A \to B$ given by Corollary 2.4 is an algebra isomorphism; either by Nagasawa's theorem or, more directly here, by the second part of Theorem 2.1 applied to the isometry $\widehat{T_1} = \Gamma_B\,T_1\,{\Gamma_A}^{-1}$ as in the proof of Proposition 2.3. $\square$

In general, $T1$ does not have to commute with each element in the image of the spectral isometry T as the following example shows.

Example 2.6. Let $A = M_2(\mathbb{C})$ and $B = M_3(A) = M_3(\mathbb{C}) \otimes M_2(\mathbb{C})$. Define $T\colon A \to B$ by

$$Ta = \begin{pmatrix} a & \ell_1(a) & 0 \\ 0 & a & \ell_2(a) \\ 0 & 0 & a \end{pmatrix},$$

where ℓ_1, ℓ_2 are linear functionals on A such that $\ell_1(1) = \ell_2(1) = 1$ and $\ell_1(x_0) \neq \ell_2(x_0)$ for some $x_0 \in A$. It is easy to verify that T preserves the spectrum of every element $a \in A$; indeed, if for example a is invertible, then the inverse of Ta is given by

$$(Ta)^{-1} = \begin{pmatrix} a^{-1} & -a^{-2}\ell_1(a) & \ell_1(a)\ell_2(a)a^{-3} \\ 0 & a^{-1} & -a^{-2}\ell_2(a) \\ 0 & 0 & a^{-1} \end{pmatrix}.$$

Clearly, T is non-unital and non-surjective. Moreover, $T1\,Tx_0 \neq Tx_0\,T1$.

We shall now turn our attention to non-surjective spectral isometries with commutative codomain; first, we look at the unital case. As we saw above, even for a proper isometry the image of an algebra may not be an algebra so we need to analyse where the multiplicativity gets lost. Once again, Novinger's theorem (Theorem 2.1) will be our main tool as it describes the action of an isometry without the assumption of surjectivity.

Suppose that $T\colon E \to C(Y)$ is a unital isometry defined on a *unital subalgebra* E of $C(X)$, where both X and Y are compact Hausdorff spaces. Suppose further that E separates the points of X. Throughout we will now denote the image of T by $F = \operatorname{im} T$ and we put $Y_T = \overline{\operatorname{ch}(F)}$, the closure of the Choquet boundary of F. By (2.1) above, we have, for all $f, g \in E$ and all $t \in \operatorname{ch}(F)$,

$$T(fg)(t) = (fg)(\varphi(t)) = f(\varphi(t))\, g(\varphi(t)) = (Tf\, Tg)(t) \tag{2.4}$$

and hence, by continuity, $T(fg)(t) = (Tf\, Tg)(t)$ for all $t \in Y_T$. It follows that $T(fg) - Tf\, Tg$ is contained in the closed ideal

$$I_T = \{k \in C(Y) \mid k(t) = 0 \text{ for all } t \in Y_T\}$$

which is nothing but the kernel of the restriction homomorphism $\rho_T\colon C(Y) \to C(Y_T)$. Therefore the composition with T is multiplicative, and we have proved the following result.

Proposition 2.7. *Let X and Y be compact Hausdorff spaces. Let T be a unital isometry from a unital subalgebra E of $C(X)$ which separates the points of X into $C(Y)$. With the above notation, $\rho_T \circ T\colon E \to C(Y_T)$ is a unital algebra homomorphism.*

Remark 2.8. With the above notation and caveats suppose that $F = \operatorname{im} T$ separates the points of Y. Then Y_T coincides with the Shilov boundary ∂F of F; cf. [**6**, Section 3.3].

By applying the Gelfand representation of commutative semisimple Banach algebras as before, we can immediately draw the following consequence for unital spectral isometries.

Proposition 2.9. *Let $T\colon A \to B$ be a unital spectral isometry between the unital commutative semisimple Banach algebras A and B. Denote by Δ_T the closure of the Choquet boundary of the image of $\Gamma_B T$ in $C(\Delta(B))$ and by $\rho_T\colon C(\Delta(B)) \to C(\Delta_T)$ the restriction homomorphism. Then $T_\rho = \rho_T \circ \Gamma_B \circ T$ is a unital algebra homomorphism from A into $C(\Delta_T)$.*

Proof. As in the proof of Proposition 2.3 we define $\widehat{T} = \Gamma_B \circ T \circ {\Gamma_A}^{-1}$ and obtain a unital isometry from $\Gamma A \subseteq C(\Delta(A))$ onto $\operatorname{im} \Gamma_B\, T \subseteq C(\Delta(B))$. By Proposition 2.7, $\rho_{\widehat{T}} \circ \widehat{T}$ is multiplicative from ΓA into $C(\Delta(B)_{\widehat{T}})$. For all $x, y \in A$ we thus obtain

$$\begin{aligned} T_\rho(xy) &= \rho_T \circ \Gamma_B \circ T(xy) = \rho_{\widehat{T}} \circ \widehat{T} \circ \Gamma_A(xy) \\ &= \rho_{\widehat{T}} \circ \widehat{T}(\hat{x}\, \hat{y}) = \rho_{\widehat{T}} \circ \widehat{T}(\hat{x})\, \rho_{\widehat{T}} \circ \widehat{T}(\hat{y}) \\ &= \rho_{\widehat{T}}\big(\widehat{T}(\hat{x})\, \widehat{T}(\hat{y})\big) = \rho_T\big(\Gamma_B \circ T(x)\, \Gamma_B \circ T(y)\big) \\ &= \rho_T \circ \Gamma_B \circ T(x)\; \rho_T \circ \Gamma_B \circ T(y) = T_\rho(x)\, T_\rho(y) \end{aligned}$$

which proves the claim. □

Finally, putting everything together, we obtain our main result.

THEOREM 2.10. *Let $T\colon A \to B$ be a spectral isometry between the unital semisimple Banach algebras A and B and suppose that B is commutative. Then there is a unitary $v \in C(\Delta_T)$ such that the mapping $a \mapsto v\,\rho_T(\Gamma_B Ta)$ is multiplicative from A into $C(\Delta_T)$, where $\Delta_T = \overline{\mathrm{ch}(\mathrm{im}\,\Gamma_B T)}$ and $\rho_T\colon C(\Delta(B)) \to C(\Delta_T)$ denotes the restriction mapping.*

PROOF. The composition $S = \Gamma_B \circ T$ is a spectral isometry into $C(\Delta(B))$. By [**8**, Lemma 2.1], S is a trace, that is, $S(xy) = S(yx)$ for all $x, y \in A$. Since B is semisimple, Γ_B is injective, and since A is semisimple, T is injective [**9**, Proposition 4.2]. As a result, A is commutative.

Without the assumption $T1 = 1$, identity (2.4) changes into

$$\widehat{T}(fg)(t) = \bar{h}(t)\,(\widehat{T}f\,\widehat{T}g)(t) \qquad \big(t \in \mathrm{ch}(\mathrm{im}\,\Gamma_B T)\big), \tag{2.5}$$

where $\widehat{T} = \Gamma_B \circ T \circ {\Gamma_A}^{-1}\colon \Gamma A \to C(\Delta(B))$ is the associated isometry as in the proof of Proposition 2.9 and $h = \widehat{T}1 \in C(\Delta(B))$ is unimodular on Δ_T, cf. Theorem 2.1. It follows that $\bar{h}\,\widehat{T}(fg) - \bar{h}\,\widehat{T}(f)\,\bar{h}\,\widehat{T}(g) \in I_T = \ker \rho_T$ for all $f, g \in \Gamma A$. Therefore, $\rho_T(\bar{h})\,\rho_T \circ \Gamma_B \circ T$ is multiplicative from A into $C(\Delta_T)$.

Put $v = \rho_T(\bar{h})$. Since multiplication by a unitary is an isometric bijection, $\mathrm{im}\,\Gamma_B T$ and $\mathrm{im}\,\bar{h}\widehat{T}$ have the same Choquet boundary in $\Delta(B)$ and Δ_T is the closure of either. It follows that $a \mapsto v\,\rho_T(\Gamma_B Ta)$ is the required algebra homomorphism. $\square$

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