Polynomial Functions and the Riesz Interpolation Property

A.W. Wickstead

Pure Mathematics Research Centre, School of Mathematics and Physics, Queen’s University Belfast, Northern Ireland.

Abstract

Building on a proof by D. Handelman of a generalization of an example due to L. Fuchs we show that the space of real valued polynomials on a non-empty set $X$ of reals has the Riesz Interpolation Property if and only if $X$ is bounded.

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1. Introduction.

Let us recall the definition of two equivalent properties of an ordered vector space $E$. It is said to have the Riesz Decomposition Property (RDP) if whenever $x, y_1, y_2 \in E$ with $0 \leq x \leq y_1 + y_2$ then there exist $x_1, x_2 \in E$ with $x = x_1 + x_2$ and $0 \leq x_k \leq y_k$ ($k = 1, 2$). We say that $E$ has the Riesz Interpolation Property (RIP), sometimes termed the Riesz Separation Property, if whenever $f_1, f_2, h_1, h_2 \in E$ with $f_1, f_2 \leq h_1, h_2$ there is $g \in E$ with $f_1, f_2 \leq g \leq h_1, h_2$. The equivalence of the RDP with the RIP may be found in Theorem 1.54 of [5]. It is simple to see that a vector lattice must have these equivalent properties. The property is of substantial importance in the theory of compact convex sets as the space of continuous affine real functions on the set has the RIP if and only if the set is a Choquet Simplex which is equivalent to the uniqueness of maximal representing measures, see [1]. In the context of ordered Banach spaces the importance is that a space with a closed normal generating cone has the RIP if and only if the dual is a...
vector lattice, [6]. Some indication of the importance of the RIP in economics may be found in [2], [3] and [4].

There are simple examples of spaces which are not vector lattices which have these properties, with the simplest probably being the spaces of sequences of reals \((a_n)\) which converge to \((a_1 + a_2)/2\). However there are not many natural examples. One important example is the space of real valued polynomial functions on a closed bounded interval in the reals, under the pointwise ordering. This ought to be part of the folk lore of the theory of ordered vector spaces, dating back as it does at least to Fuchs’ 1966 Queen’s Paper [7]. However, the result seems not to be well known at all. There are probably several reasons for this, including the relative inaccessibility of [7], the lack of a complete proof in there, and the fact that the result has not made it into any standard texts. The reason for the latter fact is undoubtedly that a complete proof is not all that short. Handelman, in [8], has given a proof of a generalization where the polynomials are restricted to a subfield of the reals. That makes the proof even less transparent than it need be and also seems to miss a few cases that need to be considered. In this note we give complete details of a variant of Handelman’s proof. In fact, we are able to show that the only important feature of the domain of the functions is that it is a bounded set of reals! We prove the single theorem:

**Theorem 1.** If \(X\) is a non-empty subset of the reals then the space, \(P(X)\), of real valued polynomial functions on \(X\) has the Riesz Interpolation Property if and only if \(X\) is bounded.

It is worth pointing out that the fact that the space of rational functions on a closed bounded interval has the RIP dates back to [12] (a complete proof may be found in [5], Example 1.56).

2. Plan of the proof.

After dealing with couple of trivial cases, the proof has three main steps. We are dealing with polynomial functions on \(X\), \(f_1, f_2 \leq h_1, h_2\). Consider the set \(T = \bigcup_{i=1,2; j=1,2} \{t \in X : f_i(t) = h_j(t)\}\), which we will see we may take to be a non-empty finite set. Figure 1, where \(f_1\) and \(f_2\) are shown as solid lines but \(h_1\) and \(h_2\) as dashed lines, shows a simple case where \(T\) consists of just two points. The first step is to show that for each \(t \in T\) there is a polynomial \(g_t\) with \(f_1, f_2 \leq g_t \leq h_1, h_2\) on a neighbourhood of \(t\) in \(X\). This is done by giving a condition on the derivatives of \(g_t\) at \(t\) which ensures this.
Figure 2 shows only parts of the graphs of two such interpolating functions on neighbourhoods of the corresponding points $t$.

The second step, illustrated in Figure 3, is to use the pigeon hole principle in the ring of polynomial functions on $X$ to show that this condition can be made to hold at all points of $T$ simultaneously. Of course the interpolation will not hold at all points of $X$. Now subtract the function that we have created, as in Figure 4.

The final step is to change the polynomials which meet up at points $t \in T$ into rational functions that do not meet up by dividing by a product $p$ of sufficiently high powers of factors $x - t$, for $t \in T$. The existence of a function that interpolates on a neighbourhood of $T$ is used to ascertain the behaviour of these rational functions near the points $t \in T$. The behaviour is shown in Figure 5, where the points $t \in T$ have become poles with very simple behaviour. There is now a uniformly strictly positive gap between the upper and lower functions and we use Weierstrass approximation to put a polynomial in between. Multiplying by the polynomial $p$ will give the interpolating polynomial that we seek.

3. Proof of the Theorem.

Notational Convention. In order to prevent repetition in the proof we make now the convention that $f, g, h, p, q, r$ (and subscripted variants) will
Figure 2: The interpolating functions on neighbourhoods of each point of $T$.

Figure 3: A function that interpolates on a neighbourhood of $T$. 
Figure 4: Original functions, shifted by the function that interpolates on a neighbourhood of $T$.

Figure 5: Strict separation after dividing by a polynomial.
always denote polynomial functions. We exclusively use greek letters for non-polynomial functions.

Let us start by removing some easy cases. Note first that we may assume without loss of generality that the set $X$ is closed as polynomials are continuous. Each of the four sets \{ $t \in X : f_i(t) = h_j(t)$ \} is the zero set of a polynomial so is finite unless $f_i = h_j$. If, for example, $f_1 = h_1$, then we have $f_2 \leq h_1 = f_1 \leq h_2$ so we can simply take $g = f_1$ for our interpolating function. If $X$ is bounded and closed and all four sets are empty, then $h_1(x) \land h_2(x) > f_1(x) \lor f_2(x)$ at all points $x \in X$. By compactness and continuity there is $\epsilon > 0$ with $h_1(x) \land h_2(x) \geq f_1(x) \lor f_2(x) + \epsilon$ for all $x \in X$. Set $\phi(x) = (h_1(x) \land h_2(x) + f_1(x) \lor f_2(x)) / 2$ and use Weierstrass approximation to produce $g \in P(X)$ with $\|\phi - g\|_\infty < \epsilon / 2$, where $\| \cdot \|_\infty$ denotes the usual supremum norm, and it is clear that $h_1(x), h_2(x) \geq h_1(x) \land h_2(x) \geq g(x) \geq f_1(x) \lor f_2(x) \geq f_1(x), f_2(x)$. We will assume for the remainder of this proof that $T$ is a non-empty finite set.

**Definition 2.** The details of the proofs to come depend on the location of a point $t$ in the set $X$. We term $t$ a left point if there is a sequence $(y_n)$ in $X \cap (t, \infty)$ with $y_n \to t$ but no sequence $(x_n)$ in $X \cap (-\infty, t)$ with $x_n \to t$ (think of the left hand endpoint of a closed interval). Similarly $t$ is a right point if there is a sequence $(x_n)$ in $X \cap (-\infty, t)$ with $x_n \to t$ but no sequence $(y_n)$ in $X \cap (t, \infty)$ with $y_n \to t$. If there are both sequences $(x_n)$ in $X \cap (-\infty, t)$ and $(y_n)$ in $X \cap (t, \infty)$ converging to $t$ then we call $t$ a mid point. The fourth, and only remaining, possibility is that $t$ is an isolated point of $X$.

We need conditions that ensure that inequalities hold on neighbourhoods of a point. We actually want the corollary that follows the next lemma, but the lemma is a useful starting point. The lemma is easily established by looking at all possible combinations of parity of $k$ and sign of $q(t)$.

**Lemma 3.** Let $\emptyset \neq X \subset \mathbb{R}$, $t \in X$ and $p$ be a non-zero polynomial on $X$ with $p(t) = 0$, so that $p(t) = (x - t)^k q(t)$ where $k \in \mathbb{N}$ and $q$ is a polynomial with $q(t) \neq 0$. There is neighbourhood $U$ of $t$ in $X$ on which $p$ is non-negative if and only one of the following holds:

1. $t$ is a mid point, $k$ is even and $q(t) > 0$.
2. $t$ is a left point and $q(t) > 0$.
3. $t$ is a right point and either
   (a) $k$ is even and $q(t) > 0$, or
4. $t$ is an isolated point.

We use the usual notation of $p^{(j)}$ for the $j$'th derivative of $p$ and make the convention that $p^{(0)} = p$.

**Corollary 4.** Let $\emptyset \neq X \subset \mathbb{R}$, $t \in X$, $p$ a non constant polynomial. Consider the sequence $\left(p^{(j)}(t)\right)_{j=0}^{\infty}$ which is eventually non-zero. Let $m$ be the smallest integer with $p^{(m)}(t) \neq 0$.

1. If $t$ is a mid point of $X$ then $p$ is non-negative on a neighbourhood of $t$ in $X$ if and only if $m$ is even and $p^{(m)}(t) > 0$.
2. If $t$ is a left point of $X$ then $p$ is non-negative on a neighbourhood of $t$ in $X$ if and only if $p^{(m)}(t) > 0$.
3. If $t$ is a right point of $X$ then $p$ is non-negative on a neighbourhood of $t$ in $X$ if and only if either
   - (a) $m$ is even and $p^{(m)}(t) > 0$, or
   - (b) $m$ is odd and $p^{(m)}(t) < 0$.

**Proof.** The sequence of derivatives is eventually non-zero as otherwise $p$ would be identically zero. If $m > 0$ the corollary follows directly from the preceding lemma by observing that a polynomial $p$ is divisible by $(x - t)^k$ if and only if $p^{(j)}(t) = 0$ for $0 \leq j < k$ and that, by Leibnitz’s theorem, if $p(x) = (x - t)^m q(x)$ then $p^{(m)}(t) = m! q(t)$. If $m = 0$ then the conclusion is immediate anyway.

Before starting the first main step in our proof let us make the following observation, which ensures that the sufficient conditions in the following proposition can actually be realised.

**Lemma 5.** If $m$ is a positive integer and $(r_j)_{j=0}^{m}$ is a sequence of reals then there is a polynomial function $p$ with $p^{(j)}(t) = r_j$ for $0 \leq j \leq m$.

**Proof.**

$$p(x) = \sum_{j=0}^{m} \frac{r_j}{j!} (x - t)^j$$

is one such polynomial. \qed
Proposition 6. If \( t \in X \) there is an integer \( m \) together with reals \( r_j \) \((0 \leq j \leq m)\) such that if the real polynomial \( g \) satisfies \( g^{(j)}(t) = r_j \) \((0 \leq j \leq m)\) then there is a neighbourhood of \( t \) in \( X \) on which \( f_1, f_2 \leq g \leq h_1, h_2 \).

Proof. We deal first with the case that \( t \) is a mid point of \( X \). We divide the proof into two cases depending on whether or not one of \( f_1 \) or \( f_2 \) dominates the other on a neighbourhood of \( t \).

Suppose first that, say, \( f_2 \leq f_1 \) on a neighbourhood of \( t \). If we can ensure that \( f_1 \leq g \leq h_1, h_2 \) on a neighbourhood of \( t \) then certainly we have \( f_1, f_2 \leq g \leq h_1, h_2 \) on some neighbourhood of \( t \). As \( f_1 \leq h_i \) \((i = 1, 2)\) on the whole of \( X \) and therefore on a neighbourhood of \( t \), Corollary 4 gives us non-negative even integers \( m_i \) with \( f^{(j)}_1(t) = h^{(j)}_i(t) \) \((-1 < j < m_i)\) and \( f^{(m_i)}_1(t) < h^{(m_i)}_i(t) \). Let \( m = \max\{m_1, m_2\} \), which is still non-negative and even. Set \( r_j = f^{(j)}_1(t) \) for \(-1 < j < m\) and choose \( r_m \) with \( f^{(m)}_1(t) < r_m < \min\{h^{(m)}_i(t) : m_i = m\} \).

If \( g^{(j)}(t) = r_j \) for \(-1 < j < m\) then Corollary 4 certainly shows that \( f_1 \leq g \) on a neighbourhood of \( t \). Certainly, we have \( g^{(j)}(t) = h^{(j)}_i(t) \) for \(-1 < j < m_i\). If \( m_i = m \) then our choice of \( r_m \) forces \( g^{(m)}(t) < h^{(m)}_i(t) \) whilst if \( m_i < m \) then we know that \( g^{(m)}(t) = f^{(m)}_1(t) < h^{(m)}_i(t) \), so that in either case we have \( g \leq h_i \) on a neighbourhood of \( t \).

Now suppose that neither \( f_1 \leq f_2 \) nor \( f_2 \leq f_1 \) on a neighbourhood of \( t \). Consider initially the one upper function \( h_1 \) and the two lower functions \( f_i \). As \( h_1 \geq f_1, f_2 \) there are non-negative even integers \( m_i \) with \( h^{(j)}_1(t) = f^{(j)}_i(t) \) for \(-1 < j < m_i \) and \( h^{(m_i)}_1(t) > f^{(m_i)}_i(t) \). If \( m_1 < m_2 \) then \( f^{(j)}_1(t) = f^{(j)}_2(t) = h^{(j)}_i(t) \) for \(-1 < j < m_1 \) whilst \( f^{(m_i)}_1(t) < h^{(m_i)}_1(t) = f^{(m_i)}_2(t) \) so that Corollary 4 shows that \( f_1 \leq f_2 \) on a neighbourhood of \( t \), which we are assuming does not hold, so we may assume that \( m_1 \geq m_2 \). Similarly we may assume that \( m_2 \geq m_1 \) and hence that \( m_1 = m_2 \). A similar argument holds for \( h_2 \), of course. There are thus non-negative even integers \( n_i \) \((i = 1, 2)\) with \( h^{(j)}_i(t) = f^{(j)}_k(t) \) for \(0 \leq j < n_i \), \( k = 1, 2 \) and \( h^{(n_i)}_i(t) > f^{(n_i)}_k(t) \), again for both \( k = 1 \) and \( k = 2 \). Let \( n = \max\{n_1, n_2\} \), \( r_j = f^{(j)}_1(t) \) for \(0 \leq j < n \) and choose \( r_n \) with \( f^{(n)}_k(t) < r_n < h^{(n)}_i(t) \) for \( i = 1, 2 \) and \( k = 1, 2 \). Now repeat the arguments at the end of the first case.

If \( t \) is a left point, then the same arguments may be used. All deductions that integers are even will be lost, but they will also not be necessary to deduce the desired local inequalities. To deal with right points, just observe that if \( t \) is a right point for the set \( X \) then \(-t \) is a left point for the set \(-X = \{-x : x \in X\}\) and the desired conclusion follows easily. Finally, for
isolated points, we take $m = 0$ and $r_0 = f_1(t) = h_1(t)$.

Although the preceding proposition applies at every point of $X$, our interest is only in the points of $T$. The next phase of the proof is to extend the interpolation to a neighbourhood of the whole set $T$. It makes use of the Chinese Remainder Theorem for rings, see for example Theorem II.2.1 of [10], which states that if $I_j (1 \leq j \leq n)$ are mutually co-prime ideals in a ring $R$ then given $r_j \in R, (1 \leq j \leq n)$ there is $r \in R$ with $r - r_j \in I_j$. Two ideals $I_1$ and $I_2$ are co-prime in $R$ if $I_1 + I_2 = R$. We apply this to the ideals $P_{t,n} = P(X) \times (x - t)^n$ in $P(X)$, for $t \in X$ and $n \in \mathbb{N}$. Equivalently, $P_{t,n}$ consists of those polynomials $p$ with $p^{(k)}(t) = 0$ for $0 \leq k < n$. If $s, t \in X$ with $s \neq t$ then it is surely well known that $P_{s,m}$ and $P_{t,n}$ are co-prime in $P(X)$, but we indicate a proof for completeness.

**Lemma 7.** If $X$ is any non-empty subset of $\mathbb{R}$, $s, t \in X$ with $s \neq t$ and $m, n \in \mathbb{N}$ then $P(X) = P_{s,m} + P_{t,n}$.

**Proof.** It is required to prove that every polynomial function $f$ can be written as $f(x) = g(x)(x - s)^m + h(x)(x - t)^n$ where $g, h \in P(X)$. It clearly suffices to prove this for $f = 1$, the constantly one function. Consider $1/((x - s)^m(x - t)^n)$. There are reals $a_i (1 \leq i \leq m)$ and $b_j (1 \leq j \leq n)$ with

$$
\frac{1}{(x - s)^m(x - t)^n} = \sum_{i=1}^{m} \frac{a_i}{(x - s)^i} + \sum_{j=1}^{n} \frac{b_j}{(x - t)^j}
$$

$$
= \frac{1}{(x - s)^m} \left[ \sum_{i=1}^{m} a_i (x - s)^{m-i} \right] + \frac{1}{(x - t)^n} \left[ \sum_{j=1}^{n} b_j (x - t)^{n-j} \right]
$$

and hence

$$
1 = (x - t)^n \left[ \sum_{i=1}^{m} a_i (x - s)^{m-i} \right] + (x - s)^m \left[ \sum_{j=1}^{n} b_j (x - t)^{n-j} \right] \in P_{t,n} + P_{s,m}.
$$

**Proposition 8.** There is a polynomial $g$ and a neighbourhood of $T$ on which $f_1, f_2 \leq g \leq h_1, h_2$.  

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\textbf{Proof.} If \( t \in T \), by Proposition 6, for each \( t \in T \) there exist an integer \( m_t \) and a sequence of reals \((r^i_j)_{j=0}^{m_t}\) such that any polynomial \( g \) with \( g(t) = r_j \), \( 0 \leq j \leq m_t \), will have \( f_1, f_2 \leq g \leq h_1, h_2 \) on a neighbourhood of \( t \). By Lemma 5 there is such a polynomial, which we label \( g_t \). By the Chinese Remainder Theorem, there is a polynomial \( g \) with \( g - g_t \in P_{t,m_t+1} \) for each \( t \in T \). But this means that \( g^{(j)}(t) = g_t^{(j)}(t) \) for \( 0 \leq j \leq m_t \) so Proposition 6 shows that \( 0, f_1 \leq g \leq h_1, h_2 \) on a neighbourhood of \( t \) for all \( t \in T \). \( \square \)

At this stage we know that if \( f_1, f_2 \leq h_1, h_2 \) then there exist a polynomial \( g \) and a neighbourhood \( U \) of \( T \) with \( f_1, f_2 \leq g \leq h_1, h_2 \) on \( U \). Let \( S \) be the points in \( T \) that are not isolated in \( X \) and \( Y = X \setminus (T \setminus S) \). For each \( t \in S \), let \( m_t \) be an even integer which strictly exceeds the greatest power of \( x - t \) that divides any of the differences \( f_1 - g, f_2 - g, h_1 - g \) and \( h_2 - g \). Let \( p = \prod_{t \in S}(x - t)^{m_t} \). As each \( m_t \) is even, \( p \) is non-negative on \( Y \) and on \( Y \) the four rational functions
\[
\frac{f_1 - g}{p}, \frac{f_2 - g}{p}, \frac{h_1 - g}{p}, \frac{h_2 - g}{p}
\]
all have poles at each point of \( S \) and nowhere else.

If \( t \in S \) then on a neighbourhood of \( t \), \( h_i \geq g \) so that \((h_i - g)/p\) is non-negative except at \( t \) itself which is a pole, so that \( \lim_{x \to t}(h_i - g)/p = +\infty \). Similarly, as \((f_i - g)/p\) is non-positive on a neighbourhood of \( t \), we have \( \lim_{x \to t}(f_i - g)/p = -\infty \). From now on it is convenient to regard these four functions as being extended real-valued functions on \( Y \), taking values in \( \{-\infty, +\infty\} \) at their poles. This convention makes them continuous for the usual topology on the extended reals. As the functions \((f_i - g)/p\) do not take the value \(+\infty\), the supremum of the values that they take on \( Y \) are (finite) reals. Let \( a_i = \sup \{(f_i - g)/p\}(Y) \in \mathbb{R} \). Similarly we may set \( b_i = \inf \{(h_i - g)/p\}(Y) \). Choose \( c, d \in \mathbb{R} \) with \( c > \max\{a_1, a_2, b_1, b_2\} \) and \( d < \min\{a_1, a_2, b_1, b_2\} \). Let \( V \) be the open subset of \( Y \) on which \( d > (f_1 - g)/p, (f_2 - g)/p \) and \((h_1 - g)/p, (h_2 - g)/p \geq c \). As each of the four functions is infinite at each point of \( S \), we certainly have \( S \subseteq V \).

On \( Y \setminus S \) we have the strict inequalities \((h_1 - g)/p(x), (h_2 - g)/p(x) > ((f_1 - g)/p(x), ((f_2 - g)/p(x) as the values in question are all real and if we had equality then we would have equality of one \( h_i(x) \) with an \( f_j(x) \) and therefore \( x \in T \). It follows from the compactness of \( X \setminus V \) that there is \( \epsilon > 0 \) with all four differences \( h_i(x) - f_j(x) \geq \epsilon \) on \( X \setminus V \). On \( V \) all four differences are least \( c - d > 0 \). Let \( \eta = \min\{\epsilon, c - d\} > 0 \). Set \( \phi(x) = \)
\[ \inf \{ ((h_1 - g)/p)(x), (h_2 - g)/p(x), c \} \text{ and } \psi(x) = \sup \{ ((f_1 - g)/p)(x), ((f_2 - g)/p)(x), d \} \text{ so that } \phi \text{ and } \psi \text{ are continuous real-valued functions on the whole of } Y \text{ with } \phi(x) \geq \psi(x) + \eta \text{ for all } x \in Y. \] Extend \phi \text{ and } \psi \text{ to the whole of } X \text{ by setting } \phi(x) = c \text{ and } \psi(x) = d \text{ for any isolated point } x. \] By the Weierstrass approximation theorem there is a polynomial \( q \) with \( \|q - (\phi + \psi)/2\|_\infty < \eta/2 \) (where \( \| \cdot \|_\infty \) denotes the usual supremum norm) and therefore we have

\[
(h_1 - g)/p, (h_2 - g)/p \geq \phi \geq \psi \geq (f_1 - g)/p, (f_2 - g)/p
\]

pointwise on \( X \). Multiplying by \( p \) and adding on \( g \) we find that

\[
h_1, h_2 \geq pq + g \geq h = f_1, f_2
\]

at least at all points of \( X \setminus S \) and therefore on the whole of \( X \) by continuity.

For the converse, suppose that \( X \) is unbounded and therefore infinite. Suppose that \( X \) is not bounded above (the proof when it is not bounded below is similar.) Choose \( x_i \in X \) (1 \( \leq i \leq 4 \)) with \( x_1 < x_2 < x_3 < x_4 \) and \( x_4 > 0 \). Consider the functions \( h_i(x) = (x - x_i)^2 \) for \( 1 \leq i \leq 3 \), \( f_1 = 0 \) and \( f_2 = \alpha(x - (x_3 + x_4)/2) \), where we will choose \( \alpha \) shortly. For \( x \leq (x_3 + x_4)/2 \) we certainly have \( h_1(x), h_2(x), h_3(x) \geq f_1(x) = 0 \geq f_2(x) \). If \( x > (x_3 + x_4)/2 \) then \( h_1(x) > h_2(x) > h_3(x) \). Consider \( h_3(x) - f_2(x) = x^2 - (2x_3 + \alpha)x + (x_3^2 + \alpha(x_3 + x_4)/2) \). The discriminant of this quadratic equation reduces to \( \alpha^2 - 2\alpha x_4 \). Choosing \( 0 < \alpha < 2x_4 \) (remember that we specified \( x_4 > 0 \)) then \( h_3 \) and \( f_2 \) are never equal on \( \mathbb{R} \) so clearly \( h_3 \geq f_2 \) on \( \mathbb{R} \). We thus have \( h_1, h_2, h_3 \geq f_1, f_2 \). If \( P(X) \) had the RIP then there would be a polynomial \( g \) with \( h_1, h_2, h_3 \geq g \geq f_1, f_2 \). As \( h_i(x_i) = f_1(x_i) = 0 \) we have \( g(x_i) = 0 \) for \( i = 1, 2, 3 \). Also, we have \( g(x) \geq f_2(x) > 0 \) for \( x > (x_3 + x_4)/2 \) so that \( g \) is not constantly zero. However, \( g \) has at least three zeros, so must be of degree 3 or higher. Also, as \( x \to +\infty \), the fact that \( g(x) \geq f_2(x) \) shows that \( g(x) \to +\infty \). As \( g \) is of degree at least 3, as \( x \to +\infty \) \( g(x) \to +\infty \) faster than, say, \( h_3(x) \to +\infty \) which is impossible if \( g \leq h_3 \).

4. Some addenda.

The proof of Theorem 1 obviously depends heavily on some very special properties of polynomials on the reals. It is not quite so immediate that the conclusion of Theorem 1 actually does fail even for bounded subsets of, say, \( \mathbb{R}^2 \). Nevertheless, it does fail, as is evidenced by the following example, Example 2 of [11], due to Nagel and Rudin.
Example 9. Take $X = [-1, 1]^2 \subset \mathbb{R}^2$. Let $f_1(x, y) = (x+y)^2$, $f_2(x, y) = (x-y)^2$ and $g(x, y) = 2x^2$, $0 \leq g \leq f_1 + f_2$ but Nagel and Rudin show that there are no polynomials $g_i$ (in fact no continuously twice differentiable functions) with $g_1 + g_2 = g$ and $0 \leq g_i \leq f_i$. Therefore the space of polynomials on $X$ also does not have the RIP.

Incidentally, in footnote 4 on page 60 of [4] it is stated that it is an open question in the mathematical literature as to whether or not there is an ordered vector space which does not have the RIP yet for which the order dual is a vector lattice. Nagel and Rudin’s example answers this question.

Example 10. The space of real polynomials on $[-1, 1]^2$, $P([-1, 1]^2)$, does not have the RIP, yet its order dual is a vector lattice.

Proof. This space is dense in $C([-1, 1]^2)$ for the supremum norm and is cofinal, since it contains the constant functions. It follows from the Krein-Rutman extension theorem (Corollary 1.6.2 of [9]) that every positive (and therefore every regular) functional on the space of polynomials extends to the space of continuous functions. Regular functionals on the Banach lattice of continuous functions are norm bounded, so the extension is unique. The positive cone in $P([-1, 1]^2)$ is dense in that of $C([-1, 1]^2)$ as if $\phi \in C([-1, 1]^2)_+$ and $\epsilon > 0$ then Weierstrass approximation gives a polynomial $p$ with $\|p - (\phi + \epsilon)\|_\infty < \epsilon$ so that $p \geq 0$ and $\|p - \phi\|_\infty < 2\epsilon$. It follows that a linear functional on $C([-1, 1]^2)$ is positive if and only if its restriction to $P([-1, 1]^2)$ is positive. Thus the order dual of $P([-1, 1]^2)$ is order isomorphic to the order dual of the Banach lattice $C([-1, 1]^2)$, which is certainly a vector lattice.

We know that at least for some rather simple subsets $X \subset \mathbb{R}$, $P(X)$ may actually be a vector lattice. This certainly holds if $X$ is a finite set as then $P(X)$ may be identified with $\mathbb{R}^n$, where $n$ is the cardinality of $X$. In fact, this is the only case when we obtain a lattice.

Proposition 11. If $X$ is a non-empty subset of $\mathbb{R}$ then $P(X)$ is a vector lattice if and only if $X$ is a finite set.

Proof. Suppose that $P(X)$ is a vector lattice, then it certainly has the RIP so that $X$ is bounded. Without loss of generality, we may assume that $X$ is closed. Observe first that the lattice operations in $P(X)$ must be pointwise ones. Indeed if $f, g \in P(X)$ have supremum $h \in P(X)$ and for some $t \in X$
we have \( h(t) - \max\{f(t), g(t)\} = \epsilon > 0 \) then consider the continuous function \( \phi(x) = \max\{f(x), g(x)\} + \epsilon/2 \). As \( X \) is bounded, Weierstrass approximation gives us \( p \in P(X) \) with \( \|p - \phi\|_\infty < \epsilon/2 \). Then \( p \geq f, g \) and \( p(t) < h(t) \), contradicting \( h \) being the supremum of \( f \) and \( g \).

If \( X \) is not finite then it will have an accumulation point, \( t \). Without loss of generality there is \( s \in X \) with \( s < t \), else consider \( P(-X) \) which is also a lattice. Let \( f(x) = (s + t) - 2x \) so that \( f(s) > 0 \) and \( f(t) < 0 \). If \( P(X) \) were a lattice then \( f^+(x) = f(x)^+ \) would be a polynomial. But \( f(x)^+ = 0 \) on a neighbourhood of \( t \) which has infinitely many points as \( t \) is an accumulation point of \( X \). Thus \( f^+ \) has infinitely many zeroes, so is identically zero. But \( f^+(s) = f(s)^+ > 0 \), which is a contradiction.

References


