VECTOR AND BANACH LATTICES

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1. Vector lattices

Definition 1.1. An *order* on a non-empty set M is a relation $\leq$ such that

(1) $x \leq x$ for all $x \in M$,
(2) $x \leq y$ and $y \leq x$ imply that $x = y$.
(3) $x \leq y$ and $y \leq z$ implies that $x \leq z$.

We use $y \geq x$ as a synonym for $x \leq y$. If A is a non-empty subset of M then x is an *upper bound* for A if $a \leq x$ for all $a \in A$. In this case we also say that A *is bounded above*. An upper bound x for A is its *least upper bound* or *supremum* if for any other upper bound y for A we have $x \leq y$. The terms *lower bound*, *bounded below*, *greatest lower bound* and *infimum* are defined analogously. Sets that are both bounded above and below are termed *order bounded*. An *order interval* in M is a set of the form $[x, y] = \{m \in M : x \leq m \leq y\}$.

A *lattice* is a non-empty set M with an order $\leq$ such that every pair of elements $x, y \in M$ has both a supremum, $x \vee y$ and an infimum $x \wedge y$. The supremum of a general subset A of M, when it exists, is denoted by any of $\sup(A)$, $\bigvee A$, $\sup\{a : a \in A\}$, $\bigvee\{a : a \in A\}$ or $\bigvee_{a\in A} a$.

Definition 1.2. An *ordered vector space* is a real vector space E which is also an ordered space with the linear and order structures connected by the implications

(1) If $x, y, z \in E$ and $x \leq y$ then $x + z \leq y + z$,
(2) If $x, y \in E$, $x \leq y$ and $0 \leq \alpha \in \mathbb{R}$ then $\alpha x \leq \alpha y$.

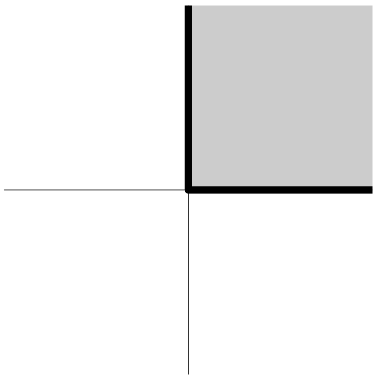
The set $E_+ = \{x \in E : x \geq 0\}$ is termed the *positive cone* in E and its elements are termed *positive* (rather than non-negative.)

An ordered vector space which is also a lattice is a *vector lattice* or *Riesz space*. As 0 is a rather special element of a vector space there are some associated special notations. The *positive part* of x is $x^+ = x \vee 0$, whilst the *negative part* (which is positive!) is $x^- = (-x) \vee 0$. The *modulus* of x is $|x| = x \vee (-x)$. We say that $x, y \in E$ are *disjoint*, written $x \perp y$, if $|x| \wedge |y| = 0$. If $A \subset E$ then $A^d = \{y : y \perp a\ \forall a \in A\}$.

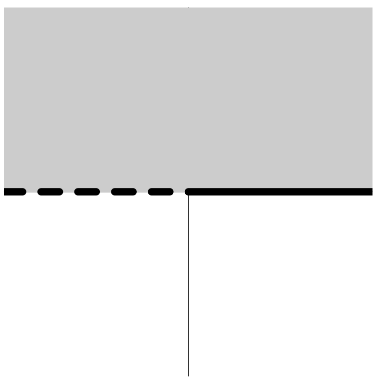
It is elementary, but often useful, that x^+ and x^- are disjoint and that $x = x^+ - x^-$ whilst $|x| = x^+ + x^-$.

Example 1.3. The most obvious example of a vector lattice is the reals with all the usual operations. The *usual* or *standard* order on $\mathbb{R}^n$ is that in which $(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n)$ means that $x_k \leq y_k$ for $k = 1, 2, \ldots n$. This order makes $\mathbb{R}^n$ into a vector lattice in which $(x_k) \vee (y_k) = (x_k \vee y_k)$ and $(x_k) \wedge (y_k) = (x_k \wedge y_k)$. Hence $(x_k)^+ = (x_k^+), (x_k)^- = (x_k^-)$ and $|(x_k)| = (|x_k|)$.

There is another order on, for example, $\mathbb{R}^2$ which has been studied namely the *lexicographic* or *dictionary* order. Under this order $(x_1, x_2) \leq (y_1, y_2)$ means that *either* $x_1 < y_1$ *or* $x_1 = y_1$ and $x_2 \leq y_2$.



Positive cone for the usual order on $\mathbb{R}^2$



Positive cone for the lexicographic order on $\mathbb{R}^2$

The lexicographic order fails to have one property that the usual order on the reals (and that on $\mathbb{R}^n$) has, namely the *Archimedean* property which states that if $nx \leq y$ for all $n \in \mathbb{N}$ then $x \leq 0$. For example $n(0, 1) \leq (1, 0)$ for all $n \in \mathbb{N}$ yet $(0, 1) \not\leq (0, 0)$. A formulation that is equivalent to a vector lattice being Archimedean is that if you consider any infinite affine line in the space (i.e. not necessarily going through the origin) then its intersection with the positive cone is closed in the usual topology of the line. Non-Archimedean vector lattices will play no further part in my talks and the term "vector lattice" may be taken as shorthand for "Archimedean vector lattice" from now on.

Example 1.4. Function spaces are important examples of Archimedean vector lattices. Let X be a non-empty set and take E to be the space of all real-valued functions on X. Order this with the pointwise order under which $f \leq g \Leftrightarrow f(x) \leq g(x)$ for all $x \in X$ and give it the pointwise vector operations and we have an ordered vector space. It is clear that if f and g are real-valued functions on X then so are the functions $\phi(x) = f(x) \vee g(x)$ and $\psi(x) = f(x) \wedge g(x)$ for all $x \in X$ and it is clear that $\phi = f \vee g$ and that $\psi = f \wedge g$, so that E is a vector lattice.

E is Archimedean as if $nf \leq g$ for all $n \in \mathbb{N}$ then $nf(x) \leq g(x)$ for all $n \in \mathbb{N}$ and for all $x \in X$. As $\mathbb{R}$ is Archimedean, it follows that $f(x) \leq 0$ for all $x \in X$ and hence that $f \leq 0$ (where this 0 is the zero function on X.)

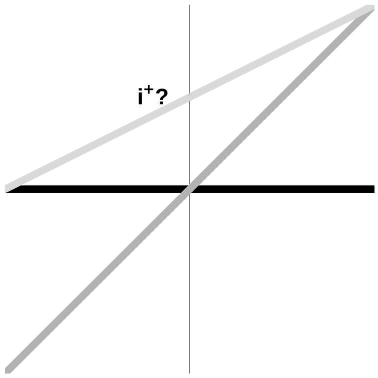
E will have many vector subspaces which are also vector lattices under the same order, for example the bounded functions; if X has a topology then we could take the continuous functions or continuous bounded functions. Many other examples will readily come to mind.

There are of course many ordered vector spaces which are not vector lattices. The space of polynomial functions on $[-1, 1]$ with the pointwise vector and order is an ordered vector space but **not** a vector lattice. What could the positive part of the identity function on $[-1, 1]$ be? It clearly isn't the pointwise supremum which isn't a polynomial. It takes a little thought and effort to prove that there is no positive part. The same is true for the space of differentiable real-valued functions on an interval.

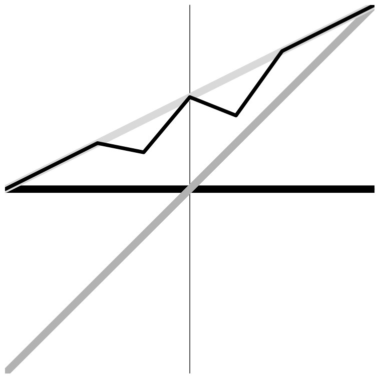
A simpler, but rather uninteresting, example of an ordered vector space which is not a vector lattice is $\{f \in C([-1, 1]) : 2f(0) = f(1) + f(-1)\}$. If $i(x) = x$ then no matter what we thought i^+ was, we will have

$$2i^+(0) = i^+(-1) + i^+(1) \geq 0 + i(1) = 1$$

so that $i(1) \geq 1/2 > 0$. We can now decrease this function slightly on intervals $(-\epsilon_1, 0)$ and $(0, \epsilon_2)$ in such a way that it remains an upper bound for both i and the zero function, provided that ϵ_1 and ϵ_2 are sufficiently small positive reals.



Can the light grey line be the graph of i^+?



No it can't!

A vector subspace of a vector lattice inherits an order from that on the vector lattice. It may or may not be a lattice in that order. For example the polynomials on $[-1, 1]$ are not a lattice even though they sit inside the lattice of all continuous real-valued functions on $[-1, 1]$. The linear functions on $[-1, 1]$ form a (two-dimensional) lattice. However the lattice operations in that subspace are not the same as the lattice operations in the whole of $C([-1, 1])$ as, for example, the positive part of i in this subspace is the linear function $x \mapsto (x + 1)/2$ (see the first picture on this page) rather than the pointwise supremum. Although this subspace is a lattice for the inherited order, it fails to be a *sublattice!*

Definition 1.5. A subset A of a lattice M is a *sublattice* if $x, y \in A$ implies that $x \vee y, x \wedge y \in A$, where these lattice operations are computed in M. A *vector sublattice* of a vector lattice is simply a vector subspace which is also a sublattice.

Example 1.6. Both c_0 and c are vector sublattices of ℓ_∞, when that has been given the pointwise vector and order operations.

Definition 1.7. An *ideal* J in a vector lattice E is a vector subspace such that $y \in J$, $x \in E$ and $|x| \le |y|$ together imply that $x \in J$.

Example 1.8. In ℓ_∞, c_0 is an ideal, but c is not as $\left|\big((-1)^n\big)\right| \le (1) \in c$ but $\big((-1)^n\big) \notin c$.

Definition 1.9. A *principal ideal* in E is an ideal J which is generated by a single element. I.e. there exists $e \in E_+$ such that $J = \{x : |x| \le ne \text{ for some } n \in \mathbb{N}\} = \bigcup_{n\in\mathbb{N}}[-ne, ne]$, which is often denoted by E_e. It can happen that there is $e \in E$ such that $E_e = E$, in which case e is termed a *strong order unit* or just *order unit* for E.

The space of continuous real-valued functions on a compact Hausdorff space K plays a very special role in the theory of Banach lattices and even (almost) in the theory of Archimedean vector lattices. This is connected with the fact that $\mathbf{1}_K$, the constantly one function on K, is an order unit for $C(K)$.

Definition 1.10. If e is an order unit for E then the expression

$$\|x\|_e = \inf\{\lambda \ge 0 : |x| \le \lambda e\}$$

is a norm on the Archimedean vector lattice E. If E were not Archimedean this would only be a semi-norm. It is referred to as the *order unit norm generated by e*. For example the supremum norm on $C(K)$ is precisely the order unit norm induced by $\mathbf{1}_K$.

Theorem 1.11 (Kakutani)**.** *If E is an Archimedean vector lattice with an order unit e then there is a compact Hausdorff space K and a linear mapping $J : E \to C(K)$ such that*

1. *J is a lattice isomorphism, i.e. it is one-to-one and for all $x, y \in E$, $Jx \vee Jy = J(x \vee y)$ and $Jx \wedge Jy = J(x \wedge y)$.*
2. *$J(E)$ is a sublattice of $C(K)$ which is dense for the supremum norm.*
3. *$Je = \mathbf{1}_K$.*
4. *$\|Jx\|_\infty = \|x\|_e$ for all $x \in E$.*

Proof. Theorem 2.1.3 of [8]. □

Definition 1.12. For every $x \in E_+$ we can carry out the Kakutani construction for E_x. If the image is always the whole of the corresponding space $C(K)$ then we say that E is *uniformly complete.*

Example 1.13. A simple example of a vector lattice that is not uniformly complete is the space of all real bounded sequences which take only finitely many different values. The constantly one order unit induces the supremum norm. If

$$x^n = (1, 1/2, 1/3, \dots, 1/n, 0, 0, \dots)$$

then (x^n) is Cauchy but not convergent to any limit in the space.

Text books on vector lattices, especially the older ones, abound in equalities and inequalities such as $|x-y| = |x \vee z - y \vee z| + |x \wedge z - y \wedge z|$ which often take quite a lot of proving. Luckily there is a simple rule of thumb which enable you to check the validity of any inequality (which includes equalities) that involves only linear and lattice operations and a finite number of elements of the vector lattice. These are termed *elementary inequalities.*

Theorem 1.14. *An elementary inequality is true in every Archimedean vector lattice if and only if it is true in the reals.*

Proof. This is proved on pages 66 and 67 of [8] for uniformly complete vector lattices using the Kakutani representation above. However, all Archimedean vector lattices may be embedded as sublattices of a uniformly complete vector lattice, namely their Dedekind completion (see below) from which the general result follows. □

The various proofs of this depend on representations of Archimedean vector lattices which means that ultimately they depend on the axiom of choice. If you aren't happy with that then you will have to seek out or provide elementary proofs of any inequality that you need. Nevertheless, it is a quick way to check the truth of an inequality.

After seeing the term *ideal*, it should come as no surprise that if J is an ideal in an Archimedean vector lattice space E then E/J can be made into a vector lattice with the usual quotient linear structure and the quotient order which says that $J + x \geq 0$ if and only if there is $y \in E_+$ such that $J + x = J + y$. However the quotient need not be Archimedean. It is precisely when J is *relatively uniformly closed* in E i.e., for all $x \in E_+$, $J \cap E_x$ is closed in E_x for the order unit norm induced by x. See [6] for details.

Definition 1.15. A *band* in a vector lattice E is an ideal J with the property that if $A \subseteq J$ and x is the supremum of A in E then $x \in J$.

Example 1.16. If I is any subset of $\mathbb{N}$ then $\{x \in \ell_\infty : x_n = 0 \ \forall n \in I\}$ is a band in ℓ_∞, but c_0 is not a band as the unit ball in c_0 has supremum (1) in ℓ_∞ which does not lie in c_0.

Definition 1.17. A *principal band* is one which is generated by a single element. If the band generated by e is the whole of E then e is a *weak order unit* for E.

Definition 1.18. A band B in E is a *projection band* if $E = B \oplus B^d$. In this case if $x \in B$ and $y \in B^d$ then $x+y \geq 0$ if and only if $x \geq 0$ and $y \geq 0$. The map that takes $x+y$ to y in this case is the *band projection* onto B denoted by P_B.

Example 1.19. The set $B = \{f \in C([0,2]) : f_{|[0,1]} \equiv 0\}$ is a band in $C([0,1])$ which is not a projection band.

In Dedekind complete vector lattices (defined below) every band is a projection band. There are many permanence properties of ideals, bands and projection bands which may be found in Chapter 3 of [7]

Definition 1.20. An *atom* a in a vector lattice E is a positive element such that $0 \leq b \leq a$ implies that b is a multiple of a. I.e. E_a is one-dimensional. If the band generated by the atoms is the whole of E then E is termed *atomic.*

Definition 1.21. A vector lattice is
(1) *Dedekind complete* if every non-empty subset which is bounded above has a supremum.
(2) *Dedekind σ-complete* if every countable non-empty subset which is bounded above has a supremum.

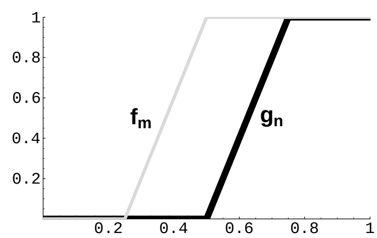
Similar statements for sets that are bounded below follow easily on multiplication by -1. Dedekind σ-complete Riesz spaces are automatically Archimedean and uniformly complete.

Example 1.22. The space $C([0,1])$ is not Dedekind σ-complete as the family of functions

$$f_m(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{m} \\ m\big(x - (\frac{1}{2} - \frac{1}{m})\big) & \frac{1}{2} - \frac{1}{m} \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

are all upper bounds for the functions

$$g_n(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ n\big(x - \frac{1}{2})\big) & \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \leq x \leq 1. \end{cases}$$



If h were the supremum of the g_n then it would lie under all the f_m's, so would be 0 on $[0, \frac{1}{2})$ and 1 on $(\frac{1}{2}, 1]$ which is incompatible with being continuous.

The space of all functions on (say) $[0,1]$ which are constant except on a countable set is Dedekind σ-complete but not Dedekind complete.

The $L^p(\mu)$ spaces are Dedekind complete except for some rather extreme examples which are not even Banach spaces.

Some of the most interesting spaces are the $C(K)$ spaces. The following result was originally due to Nakano.

Theorem 1.23. *If K is a compact Hausdorff space then*

(1) *$C(K)$ is Dedekind complete if and only if K is* Stonean*, i.e. the closure of every open subset of K is open.*
(2) *$C(K)$ is Dedekind σ-complete if and only if K is* quasi-Stonean*, i.e. the closure of every open F_σ-subset of K is open.*

Proof. Theorems 2.1.4 and 2.1.5 of [8]. □

Combined with the Kakutani representation for principal ideals in uniformly complete vector lattices this yields the following extremely useful characterisations, due to Veksler and Gejler in [10].

Theorem 1.24. *If E is a uniformly complete vector lattice then,*

(1) *E is Dedekind complete if and only if every disjoint subset of E_+ which is bounded above has a supremum.*
(2) *E is Dedekind σ-complete if and only if every disjoint countable subset of E_+ which is bounded above has a supremum.*

Although not every Archimedean vector lattice E is Dedekind complete it does have a unique *Dedekind completion* F, [7], Theorem 32.5. This is a Dedekind complete vector lattice which contains a lattice isomorphic copy of E, $\hat{E}$, (obvious definition) such that for every $y \in F$,

$$y = \sup\{\hat{x} \in \hat{E} : \hat{x} \leq y\} = \inf\{\hat{x} \in \hat{E} : \hat{x} \geq y\}.$$

There are various classes of linear mappings between vector lattices which are natural to study.

Definition 1.25. If E and F are vector lattices then

(1) $T : E \to F$ is *positive* if $x \geq 0 \Rightarrow Tx \geq 0$. The positive operators are closed under addition and multiplication by non-negative reals but not under multiplication by negative reals. The collection of all of them is denoted by $\mathcal{L}(E, F)_+$.

(2) The linear span of the positive operators is the linear space of *regular operators*, denoted by $\mathfrak{L}^r(E,F)$. This is ordered by the relation $S \geq T \Leftrightarrow S - T \in \mathfrak{L}(E,F)_+$. Clearly this definition makes $\mathfrak{L}^r(E,F)_+ = \mathfrak{L}(E,F)_+$.

(3) The *order bounded* operators are those for which the image of every order bounded subset of E is an order bounded subset of F, which is denoted by $\mathfrak{L}^b(E,F)$. It is ordered in the same way as $\mathfrak{L}^r(E,F)$.

Every regular operator must be order bounded, but the converse is false. One such may be found in Example 1.16 of [4]. Much more important is the fact that there are many cases known when they are equal.

Theorem 1.26. *If F is Dedekind complete then for every Archimedean Riesz space E $\mathfrak{L}^b(E,F) = \mathfrak{L}^r(E,F)$ is a Dedekind complete vector lattice. Furthermore if $x \in E_+$ then*

(1) $T^+(x) = \sup\{Ty : 0 \leq y \leq x\}$,
(2) $T^-(x) = \inf\{Ty : 0 \leq y \leq x\}$, *and*
(3) $|T|(x) = \sup\{|Ty| : |y| \leq x\}$,

for all $T \in \mathfrak{L}^r(E,F)$.

Proof. Theorem 1.3.2 of [8]. □

These formulae for T^+, T^- and $|T|$ are known as the *Riesz-Kantorovich* formulae. This result does not exhaust the cases where equality is known. For example in a Banach lattice setting a similar result is known if F is only assumed to be Dedekind σ-complete provided that E is separable.

There are some very special kinds of operator that come into work in this field.

Definition 1.27. If E and F are vector lattices then a linear operator $T : E \to F$ is a *lattice homomorphism* if $Tx \vee Ty = T(x \vee y)$ for all $x, y \in E$. The analogous result for infima follows automatically.

These really are a very special kind of operator.

Theorem 1.28. *If K and L are compact Hausdorff spaces and $T : C(K) \to C(L)$ is a linear lattice homomorphism then there is $w : L \to \mathbb{R}_+$ which is continuous on $D = \{k \in L : w(k) > 0\}$ and $\pi : D \to K$ which is continuous such that $Tf(k) = w(k)f(\pi k)$.*

Proof. Theorem 3.2.10 of [8]. □

Extension theorems are important in any setting and there are some important ones here. We pick out only two of the more important ones.

Definition 1.29. A linear subspace H of a vector lattice E is *majorizing* if for every $x \in E$ there is $y \in H$ such that $x \leq y$.

Theorem 1.30 (Kantorovich)**.** *If F is a Dedekind complete vector lattice, H is a majorising subspace of E and $T : H \to F$ is a positive linear operator then there is a positive linear extension of T to the whole of E.*

Proof. [8] Corollary 1.5.9. ☐

Theorem 1.31 (Luxemburg-Schep)**.** *If F is a Dedekind complete vector lattice, H is a majorising sublattice of E and $T : H \to F$ is a linear lattice homomorphism then there is a linear lattice homomorphism extending T to the whole of E.*

Proof. [8] Theorem 1.5.15. ☐

Definition 1.32. As the reals form a Dedekind complete vector lattice, we have notions of order bounded and regular functionals on E which now are known to coincide. They form the *order dual* of E, denoted by $E^\sim$, which is a Dedekind complete vector lattice under this dual ordering.

Theorem 1.33. *If $x \in E$ then the map $\hat{x} : f \mapsto f(x)$ lies in $(E^\sim)^\sim$ and $x \mapsto \hat{x}$ is a lattice homomorphism. If $E^\sim$ separates the points of E then $x \mapsto \hat{x}$ is one-to-one.*

Proof. Proposition 1.4.5 of [8]. ☐

Sometimes this *bidual* is a more convenient Dedekind complete space in which to embed the original space than its Dedekind completion. Unfortunately there are examples, e.g. $L^p([0,1])$ for $0 < p < 1$, of Dedekind complete vector lattices for which the order dual consists only of the zero functional. However, we will shortly see that for Banach lattices the order dual coincides with the Banach dual so that the Hahn-Banach theorem may be invoked to guarantee that $E^\sim$ does separate the points of E.

2. Banach lattices

Definition 2.1. A *normed lattice* is a normed space which is also a vector lattice in which $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$. A normed lattice which is also a Banach space is termed a *Banach lattice.*

Example 2.2. All of the classical (real) Banach spaces, $\ell_p, c_0, c, C(K)$, $L^p(\mu)$ etc are Banach lattices for their usual norm and the pointwise $\big($almost everywhere in the case of $L^p(\mu)\big)$ order.

Even as vector lattices, Banach lattices must be rather nice. For example they must be uniformly complete, so that several results from the first section may be invoked—especially Theorem 1.24. Once we bring the norm into play in an explicit way then things really start to become interesting.

Theorem 2.3. *In any Banach lattice the maps* $x \mapsto x^+, x \mapsto x^-, x \mapsto |x|, (x, y) \mapsto x \vee y$ *and* $(x, y) \mapsto x \wedge y$ *are all uniformly continuous.*

Proof. Proposition 5.2 of [9] □

Theorem 2.4. *If* E *is a Banach lattice and* F *a normed lattice then every regular operator from* E *into* F *is norm bounded.*

Proof. This is proved for positive operators in [8], Proposition 1.3.5, from which it follows easily for regular operators. □

Corollary 2.5. *If* E *is a Banach lattice then* $E^\sim = E^*$.

Sublattice and ideals in Banach lattices need not be closed, although the closed ones are most commonly used. Ideals that are norm closed must relatively uniformly closed so the corresponding quotient is Archimedean. In fact it will be a Banach lattice in a natural manner. Bands, on the other hand, must be closed. As far as further applications are concerned, there are several important classes of Banach lattice that need to be defined. Although this list may look rather intimidating, this list pretty much exhausts the properties that you are likely to come across, unlike Banach space theory where new properties seem to be produced in every paper.

Definition 2.6. A Banach lattice E in which $\|x \vee y\| = \|x\| \vee \|y\|$ for every $x, y \in E_+$ is termed an *AM-space*. A norm satisfying this equality is termed an *M-norm*.

Theorem 2.7 (Kakutani)**.** *For every AM-space* E *there is a compact Hausdorff space* K *and an isometric lattice homomorphism* J *of* E *onto a sublattice of* $C(K)$*. Furthermore there is a family of triples* $(k_1^i, k_2^i, \lambda^i)$ *for* $i \in I$ *such that the sublattice* JE *is precisely the space* $\{f \in C(K) : f(k_1^i) = \lambda^i f(k_2^i)\ \forall i \in I\}$.

Proof. Although partial proofs may be found in many texts, none that I know of give all the details which may be found in the original proof in [5]. □

There are many rather technical equivalences that we will not go into here, but there are some interesting *isomorphic* equivalences that are worth stating.

Theorem 2.8. *The following conditions on a Banach lattice E are equivalent:*

1. *There is an equivalent M-norm on E.*
2. *Every relatively compact subset of E has a supremum.*
3. *Every disjoint sequence in E which converges to zero in norm is order bounded in E^{**}.*
4. *There is a constant $K > 0$ such that*
$$\|x_1 + x_2 + \cdots + x_n\| \leq K \max\{\|x_1\|, \|x_2\|, \ldots, \|x_n\|\}$$
whenever $x_1, x_2, \ldots, x_n$ are disjoint in E_+.

Proof. [8], Theorem 2.1.12. □

Definition 2.9. A Banach lattice E in which $\|x + y\| = \|x\| + \|y\|$ for every $x, y \in E_+$ is termed an *AL-space.* A norm satisfying this equality is terms an *L-norm.*

Theorem 2.10 (Kakutani)**.** *If E is an AL-space then there is a Baire measure μ on a Hausdorff topological space X such that E is isometrically order isomorphic to the whole of $L^1(\mu)$.*

Proof. Theorem 2.7.1 of [8]. □

AM-spaces and AL-spaces are mutually dual, in a sense anyway.

Theorem 2.11. *If E is a Banach lattice then*

1. *E is an AL-space if and only if E^* is an AM-space, and in this case E^* has an order unit norm.*
2. *E is an AM-space if and only E^* is an AL-space.*

Proof. Page 188 of [4]. □

If one merely assumes that E is a nice enough ordered Banach space then the first assertion remains true, but the second fails.

There are several generic properties that Banach lattices may or may not possess.

Definition 2.12. The norm in a Banach lattice E is *order continuous* if, whenever $A \subset E_+$ is downward directed (i.e. if $a_1, a_2 \in A$ then there is $a_3 \in A$ with $a_3 \leq a_1, a_2$) with $\bigwedge A = 0$, we have $\bigwedge\{\|a\| : a \in A\} = 0$.

Example 2.13. If Σ is a locally compact Hausdorff space and $C_0(\Sigma)$ is the space of all continuous real-valued functions which vanish at infinity then $C_0(\Sigma)$ has an order continuous norm if and only if Σ is a discrete space.

If $1 \leq p \leq \infty$ and μ is a measure such that $L^p(\mu)$ is infinite dimensional then $L^p(\mu)$ has an order continuous norm if and only if $p < \infty$.

There are many characterizations known of order continuity of the norm. We gather some of them here into a single statement.

Theorem 2.14. *The following conditions on a Banach lattice E are equivalent:*

(1) *E has an order continuous norm.*
(2) *E is Dedekind complete and if (x_n) is a sequence in E with $x_n \downarrow 0$ then $\|x_n\| \downarrow 0$.*
(3) *E is Dedekind σ-complete and if (x_n) is a sequence in E with $x_n \downarrow 0$ then $\|x_n\| \downarrow 0$.*
(4) *Every monotone order bounded sequence in E is convergent.*
(5) *Every disjoint order bounded sequence in E_+ converges in norm to 0.*
(6) *E is an ideal in E^{**}.*
(7) *Every order interval in E is weakly compact.*
(8) *E is Dedekind σ-complete and no sublattice of E is isomorphic to ℓ_∞.*
(9) *For every sublattice H of E which is isomorphic to c_0 there is a positive linear projection of E onto H.*
(10) *Every closed ideal in E is a band.*
(11) *Every closed ideal of E is the range of a positive linear projection.*
(12) *Every band projection in E^* is weak*-continuous.*
(13) *Every band in E^* is weak*-closed.*

Proof. [8], Theorem 2.4.2, Corollary 2.4.3, Corollary 2.4.4 and Corollary 2.4.7. ☐

It is not permissible to drop Dedekind σ-completeness from either (3) or (8). $C\big(\beta(\mathbb{N})\setminus\mathbb{N}\big)$ is a counterexample for (3) whilst, for example, $C([0,1])$ is a counterexample for (8).

There are similar characterizations of when E^* has an order continuous norm. Before introducing them we give one more important definition.

Definition 2.15. A *KB-space* is a Banach lattice in which every monotone norm bounded sequence is convergent. "KB" here stands for "Kantorovich-Banach".

KB-spaces must have an order continuous norm, but c_0 has an order continuous norm without being a KB-space as the increasing sequence (u_n) is norm bounded but not convergent, where u_n is the sequence starting with n 1's and then all zeros.

Theorem 2.16. *The following conditions on a Banach lattice E are equivalent:*

(1) *E^* has an order continuous norm.*
(2) *E^* is a KB-space.*
(3) *Every disjoint norm bounded sequence in E is weakly convergent to 0.*
(4) *E^* contains no sublattice isomorphic to ℓ_∞.*
(5) *E^* contains no sublattice isomorphic to c_0.*
(6) *E contains no sublattice isomorphic to ℓ_1.*

Proof. Theorem 2.4.14 of [8]. □

Putting together the last two results we may characterize reflexive Banach lattices.

Theorem 2.17. *The following conditions on a Banach lattice E are equivalent:*

(1) *E is reflexive*
(2) *E and E^* are KB-spaces.*
(3) *E does not contain any sublattice isomorphic either to c_0 or to ℓ_1.*
(4) *E does not contain any subspace isomorphic either to c_0 or to ℓ_1.*

Proof. Theorem 2.4.15 of [8]. □

(1) and (4) are *not* equivalent for Banach spaces. This is the first result that we have met that emphasizes how special Banach lattices are even as Banach spaces. For example, even in the area of reflexivity, it is not possible for E^{**}/E to be separable without E being reflexive provided E is a Banach lattice. There are many examples of Banach spaces where this fails, such as the James space.

Banach lattices which are atomic as well as having an order continuous norm are very special.

Theorem 2.18. *the following conditions on a Banach lattice E are equivalent:*

(1) *E is atomic and has an order continuous norm.*
(2) *The linear span of the atoms is norm dense in E.*
(3) *Every order interval in E is norm compact.*

Proof. There is no easily accessible proof in a textbook at present. A full proof may be found in Theorem 5 of [11], although modern methods allow a much shorter proof. □

There are several related properties that have unfortunately been known by several names at various times. Even more unfortunately these names have been used for different properties by different authors. If you read anything using these terms, check the definitions used carefully!

Definition 2.19. Let E be a Banach lattice.

(1) The norm is *Fatou* if whenever $A \subset E_+$ is upward directed to $m \in E$ we have $\|m\| = \sup\{\|a\| : a \in A\}$.
(2) The norm is *sequentially Fatou* if whenever (a_n) is an upward sequence with supremum $m \in E$ we have $\|m\| = \sup\{\|a_n\| : n \in \mathbb{N}\}$.
(3) The norm is *weakly Fatou* if there is a constant K such that whenever $A \subset E_+$ is upward directed to $m \in E$ we have $\|m\| \leq K \sup\{\|a\| : a \in A\}$.
(4) The norm is *weakly sequentially Fatou* if there is a constant K such that whenever (a_n) is an upward sequence with supremum $m \in E$ we have $\|m\| \leq K \sup\{\|a_n\| : n \in \mathbb{N}\}$.

Notice that if $0 \leq a \leq m$ then $|a| \leq |m|$ so that $\|a\| \leq \|m\|$ so we always have, for example, $\|m\| \geq \sup\{\|a\| : a \in A\}$.

Example 2.20. Order continuous norms must be Fatou for then increasing families converge in norm to their supremum.

Let $\|\cdot\|_n$ be defined on ℓ_∞ by $\|(x_k)\|_n = \|x\|_\infty \vee n \limsup |x_k|$. If $0 \leq A \uparrow m$ then $m_k \leq \sup\{a_k : a \in A\} \leq \sup\{\|a\|_n : a \in A\}$ so that $\|m_k\|_n \leq n \sup\{\|a\|_n : a \in A\}$. However if u^m is the sequence starting with n 1's then having all zeros, then $u^m \uparrow (1)$, each $\|u^m\|_n = 1$ whilst $\|(1)\|_n = n$. Thus $\|\cdot\|_n$ is weakly Fatou but not Fatou.

If we denote ℓ_∞ with this norm by X_n then the space $\ell_\infty(X_n)$, consisting of all sequences (x_n) with $x_n \in X_n$ and with $(\|x_n\|_n)$ bounded is, in a natural way, a Banach lattice with a norm that is not weakly Fatou.

For Banach lattices satisfying a sufficient order theoretic completeness condition there are useful characterizations involving only disjoint elements.

Theorem 2.21. *A Dedekind complete Banach lattice E has a Fatou norm if and only if for every pairwise disjoint subset $A \subset E_+$ with $\bigvee A = m$ we have $\|m\| = \sup\{\|\sum_{a\in F} a\| : F \subset A\}$, where F ranges over all the finite subsets of A.*

Proof. Theorem 2.1 of [1]. □

There are analogous results for the other variants of the notion.

Definition 2.22. The norm in a Banach lattice is *Levi* if every norm bounded upward directed set of positive elements has a supremum. The norm is *sequentially Levi* if this holds for norm bounded upward directed sequences.

If E has a Levi (resp. sequentially Levi) norm then it must be Dedekind complete (resp. σ-complete.) A Banach lattice with an order continuous norm has a Levi norm if and only if it is a KB-space. A significant result, originally due to Amemiya, is:

Theorem 2.23. *A Banach lattice with a Levi norm has a weakly Fatou norm.*

Proof. Proposition 2.4.19 (i) of [8], but be aware the Meyer-Nieberg uses the term *monotonically complete* instead of *Levi.* □

Without any assumption of order completeness we have a characterization using disjoint elements only.

Theorem 2.24. *A Banach lattice E has a (sequentially) Levi norm if and only if for every pairwise disjoint (countable) set $A \subset E_+$, such that the collection of all finite sums from A is norm bounded, the set A has a supremum.*

Proof. Theorems 2.3 and 2.4 of [2]. □

The space of regular operators need not be well-behaved for the operator norm, but there is another noteworthy norm on it.

Definition 2.25. If $T \in \mathcal{L}^r(X,Y)$ then the *regular* norm of T is $\|T\|_r = \inf\{\|S\| : \pm T \leq S\}$.

No matter what the order structure of $\mathcal{L}^r(X,Y)$ we have:

Theorem 2.26. *If X and Y are Banach lattices then $\mathcal{L}^r(X,Y)$ is a Banach space under the regular norm.*

Proof. Proposition 1.3.6 of [8]. □

If $\mathcal{L}^r(X,Y)$ is a vector lattice then it is a Banach lattice under the regular norm, even though we do not know formulae for the lattice operations in general.

To finish with, I wish to point out that although Banach lattices are naturally real objects there are occasions when it is desirable to deal with complexifications of them, for example when considering spectral theory.

If E is a real Banach space then $E_{\mathbb{C}}$ may be defined as $E \oplus \imath E$ in the usual way. It turns out that if $x, y \in E$ then the expression

$$\sup\{\cos(\theta)x + \sin(\theta)y : \theta \in [0, 2\pi]\}$$

always exists in E. We define this to be the modulus of $z = x + \imath y$ and set $\|z\|_{\mathbb{C}} = \big\||z|\big\|$. This makes $\|\cdot\|_{\mathbb{C}}$ be a complex norm on $E_{\mathbb{C}}$ that extends the original norm. In the case of the classical Banach lattices this gives the obvious complex analogue of the real norm.

If $T : E \to F$ then we may define $T_{\mathbb{C}} : E_{\mathbb{C}} \to F_{\mathbb{C}}$ by $T_C(x + \imath y) = Tx + \imath Ty$. If T is bounded then $\|T\| \leq \|T\|_{\mathbb{C}} \leq 2\|T\|$, whilst if T is positive then $\|T\|_{\mathbb{C}} = \|T\|$. Strict inequality is possible in the non-positive case even for 2-dimensional Banach lattices.

Details of these constructions may be found either in §3.2 of [3] or in §2.2 of [8].

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