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NETWORK EQUILIBRIUM: OPTIMIZATION FORMULATIONS WITH BOTH QUANTITIES AND PRICES AS VARIABLES

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Abstract—The optimization formulation of the traffic assignment problem is usually stated in terms of flow (quantity) variables but can also be stated entirely in terms of price variables (the dual formulation), and recently it has also been stated in terms of a combination of both quantity and price variables. Here we consider properties and problems associated with the latter optimization formulations and relate these formulations to the traffic equilibrium conditions and to the purely quantity formulation.

1. INTRODUCTION†

In the usual optimization formulation of the traffic assignment problem, developed and presented by Beckmann *et al.* (1956), Dafermos (1971), Florian (1976), and others, all the variables represent flows on arcs or paths: see the recent useful surveys by Fernandez and Freisz (1983) and Friesz (1985). We will refer to this formulation as the quantity-quantity formulation, since both the travel demand side and the travel time or cost side of the objective function are expressed in terms of flow (quantity) variables. There is also an equivalent dual formulation of the traffic assignment problem [Carey (1985), Fukushima (1984)] in which all variables represent travel times or costs for arcs or paths. We will refer to this as the price-price formulation. If the travel demand equations, and the arc travel/cost equations, are invertable then the solution of the quantity-quantity formulation can be obtained directly from the solution of the price-price formulation, and vice-versa.

As well as the above two optimization formulations we can construct two less well-known optimization formulations, which we will refer to as a quantity-price formulation and a price-quantity formulation. A quantity-price formulation is presented in Aashtiani (1979) Ch. 5 and in the survey paper Fernandez and Friesz (1983), pp. 164–165. In the quantity-price formulation the cost part of the objective function is stated in terms of quantity variables and the demand part of the objective function is stated in terms of price variables. The quantity variables are then related to the price variables by explicitly including travel demands as functions of prices in the constraints. A price-quantity formulation can be stated analogously, the cost part of the objective function being stated in terms of price variables, the demand part in terms of quantity variables, and the price and quantity variables related to each other by including the arc travel time/cost equations explicitly in the constraints.

The purpose of the present paper is to set out some properties and difficulties associated with the quantity-price and price-quantity formulations, and in particular to relate these models to the traffic equilibrium conditions, and to the quantity-quantity formulations. In the recent research literature, variational inequality formulations of the traffic equilibrium problem have generated much more interest than optimization formulations, since the former are more recent and more general formulations of the equilibrium problem than are the latter. However, it is still important to consider problems peculiar to optimization formulations since:

a. When an optimization formulation is possible (when the Jacobian matrices of the travel demand functions and travel cost functions are symmetric) then it has great computational advantages over the variational inequality formulation. [See for example Fisk and Boyce (1983)].

b. Even when the variational inequality problem cannot be reduced to a single equivalent optimization formulation (the asymmetric case), it may be solved by solving a sequence of variational inequality problems each of which reduces to an optimization problem. Thus even in this case the various optimization formulations are important.

†I wish to thank anonymous referees for helpful comments and corrections.

c. The optimization formulation is, by definition, always applicable in the case of system optimal (as opposed to user equilibrium) traffic assignment.

In Section 2 below we set out the multi-modal quantity-price transportation equilibrium optimization model from Fernandez and Friesz (1983) and note that the nonconvexity of the constraint set makes it difficult to show directly that the optimal solution is either necessary or sufficient for a solution of the traffic equilibrium conditions. In Section 3 we circumvent this difficulty, by showing that the quantity-price model has the same soution set as the (well-behaved) quantity-quantity model, hence the former is equivalent to the same traffic equilibrium conditions as the latter. Since the quantity-quantity model is better behaved than the quantity-price model (its constraints are linear and it is convex under a weaker set of assumptions) this raises questions as to the usefulness of the latter (the quantity-price model), however, Section 4 identifies a role for the quantity-price model, for situations in which finding the inverse of the demand functions is costly, difficult or impossible. Section 5 sets out a price-quantity model analogous to the quantity-price model and shows that, with appropriate changes, Sections 2–4 below also apply to this price-quantity model. Solution techniques for the quantity-price and price-quantity models are discussed in Section 6.

2. THE QUANTITY-PRICE OPTIMIZATION MODEL

Following the formulation in Fernandez and Friesz (1983), pp. 164–165, the multicommodity transportation equilibrium problem can be formulated as an optimization problem having quantity variables (f's) on the supply side and price variables (u's) on the demand side, as in P1 below. (The arc-chain formulation of the conversation eqn (1.2–1.4) can easily be replaced by the alternative arc-node formulation, which does not require path enumeration, but this is not important here.)

P1: Minimize

$$\int_0^f c(x) \cdot \mathrm{d}x - u \cdot T(u) + \int_{u(0)}^{u} T(y) \cdot \mathrm{d}y \qquad (1.1)†$$

subject to

$$\Sigma_p \, \delta^k_{ap} h^k_p = f^k_a \qquad \forall \, (a, k) \qquad (1.2)$$

$$\Sigma_{p \epsilon P_w} h^k_p = T^k_w(u) \qquad \forall \, (w, k) \qquad (1.3)$$

$$h^k_p \geq 0 \qquad \forall \, (p, k) \qquad (1.4)$$

$$u^k_w \geq 0 \qquad \forall \, (w, k) \qquad (1.5)$$

where a = a member of L, the set of all directed links; w = a member of W, the set of all origin-destination (O-D) pairs; p = a member of P, the set of all perceived origin to destination paths; k = a member of K, the set of all user classes, which includes all transportation modes; P_w = the set of all paths joining O-D pair w; $\delta^k_{ap} = 1$, if link a lies on path p and allows traffic of class k; $\delta^k_{ap} = 0$ otherwise; f^k_a = flow of class k on link a; T^k_w = flow of class k between O-D pair w; h^k_p = flow of k on path p; c^k_a = the average travel cost experienced by class k on link a. (For a system of optimizing model rather than a user equilibrium model, c^k_a is the social marginal cost rather than the average cost.) u^k_w = perceived marginal benefit or willingness-to-pay for a trip between O-D pair w by class k. The full vectors corresponding to the above variables are $\mathbf{f} = [f^k_a]$, $\mathbf{T} = [T^k_w]$, $\mathbf{h} = [h^k_p]$, $\mathbf{c} = [C^k_a]$ and $\mathbf{u} = [u^k_w]$. The travel demand functions are $T^k_w = T^k_w(u)$, $\forall \, (w, k)$, and the travel cost functions are $c^k_a = c^k_a \, (f)$, $\forall \, (a, k)$,

†There is a slight misprint in the expression corresponding to (1.1) in the Fernandez and Friesz paper: there the − should be + and the + should be −. Also, we will omit vector transpose signs throughout below since they should be apparent from the context.

so that $\int_0^f c(x) \cdot dx \equiv \int_0^f \Sigma_{a,k} c_a^k(x) \, dx_a^k$ denotes the line integral of $c(x)$ along any path from 0 to **f**, and similarly for $\int \mathbf{T}(y) \cdot dy$. The authors also employ the following assumptions:

(i) $|K| > 1$, where $|K|$ is the number of elements in the set K.
(ii) $c_a^k = c_a^k(f)$ is a function of the full vector of flows **f**.
(iii) $\mathbf{J}(c)$, the Jacobian matrix formed from the cost functions $c(f)$, is symmetric.
(iv) $\mathbf{J}(c)$ is positive definite.
(v) $T_w^k = T_w^k(u)$ is a function of the full vector of travel prices/costs, **u**.
(vi) $\mathbf{J}(T)$, the Jacobian matrix formed from the demand functions $T(u)$, is symmetric.
(vii) $-\mathbf{J}(T)$ is positive definite.
(viii) $-\partial T_w^k / \partial u_v^\ell \leq 0$, for $(k, w) \neq (\ell, v)$.
(ix) $-\mathscr{H}(T_w^k)$, the negative of the Hessian matrix formed from the demand functions, is positive definite for each (k, w).

Assumptions (i), (ii) and (v) are introduced in order to make the model more general. Assumptions (iii) and (vi) are introduced to ensure that the first derivatives $\partial / \partial f$ and $\partial / \partial u$ respectively of the two integral terms in the objective function are $\mathbf{c}(f)$ and $\mathbf{T}(u)$. Assumptions (iv), (vii) and (ix) are introduced as sufficient, though not necessary, conditions to ensure that the objective function is convex [see Aashtiani (1979), Theorem 5.1]. Assumption (viii) enables Aashtiani to show that the Kuhn-Tucker conditions associated with P1 reduce to the following well-known multi-modal traffic equilibrium conditions.

$$(1.2\text{–}1.5) \tag{2.1}$$

$$\Sigma_a \delta_{ap}^k \, c_a^k(f) \geq u_w^k \qquad \forall \, (p, w, k) \tag{2.2}$$

$$[\Sigma_a \delta_{ap}^k \, c_a^k(f) - u_w^k] \, h_p^k = 0 \qquad \forall \, (p, w, k) \tag{2.3}$$

Solving (2.1–2.3) is of course the motivation behind program P1. Aashtiani seeks to show that the solution set of P1 is equivalent to the solution set of (2.1–2.3), but acknowledges (Aashtiani, p. 84) that there is a step still missing in the proof. He shows that the Kuhn-Tucker (K–T) conditions corresponding to P1 are identical to (2.1–2.3), but does not show that these conditions are equivalent to (i.e. necessary and sufficient for) an optimum of P1.

The usual way of showing that the K–T conditions are *sufficient* to ensure an optimum (minimum) of a mathematical program is to show that the program has a convex constraint set, and a convex of pseudoconvex objective function. Unfortunately the constraints (1.3) are nonlinear equalities and hence represent a *non*convex set.

The usual way of showing that K–T conditions *necessarily* hold at an optimum of a mathematical program is to show that the constraints of the program satisfy a "constraint qualification" (CQ) [Bazaraa and Shetty (1979)]. However, no CQ has yet been presented for program P1. The two best known and most widely applicable constraint qualifications are (a) a CQ is satisfied if the constraint set is convex and contains a strictly interior point. Neither of these CQ's are satisfied in the case of program P1, because of the nonlinear equality constraints (1.3).

Despite the above, we can in fact show that the conditions (2.1–2.3) *are* necessary and sufficient to ensure an optimum of P1. This we do in the next section (Lemmas 1, 3 and 4).

It might be thought that computational and analytic difficulties associated with P1 could be overcome by transforming program P1 into a convex program, by changing the nonlinear equality constraint (1.3) to an inequality, thus,

$$\Sigma_{p \in P_w} h_p^k \geq T_w^k(u) \qquad \forall \, (w, k) \tag{1.3'}$$

(Note that the inequality must be $\geq$ rather than $\leq$, since the latter would immediately yield the trivial optimal solution, $h = 0$, $f = 0$, $u = 0$.) Unfortunately, assumption (ix) makes (1.3′) a nonconvex set [Bazaraa and Shetty (1979), Theorems 3.3.6 and 3.5.2], unless $T(u)$ happens to be linear in u. It is, however, possible to show that if program P1 includes (1.3′) rather than

(1.3) then (a) the new program satisfies the "reverse convex" constraint qualification [defined in Mangasarian (1969), p. 103], so that the K–T conditions necessarily hold, and (b) in any optimal solution all members of (1.3′) are strict equalities, so that the solution is identical to solving the original program P1. But this approach does not enable us to show that the K–T conditions are *sufficient* for an optimum, and is redundant here in view of the simplicity of the approach which we adopt, in Lemmas 1, 3, and 4 below.

3. EQUIVALENCE OF THE QUANTITY-PRICE MODEL AND THE TRAFFIC EQUILIBRIUM CONDITIONS

To show that, given assumptions (i–vii), the solution of the quantity-price mode P1 is identical to the solution of the traffic equilibrium conditions (2.1–2.3) we proceed as follows.

a. Show that the solution set of P1 is equivalent to the solution set of the well-known quantity-quantity model P2 below (Lemma 1).

b. Show that the solution set of P2 is identical to the solution set of the equilibrium conditions (2.1–2.3) (Lemma 3).

c. The desired result follows (Lemma 4) on combining (a) and (b).

Let us now introduce the well-known quantity-quantity optimization formulation of the multi-modal equilibrium problem, using the same notation as above, thus,

$$\text{P2: Minimize} \quad \int_0^f c(x) \cdot \mathrm{d}x - \int_0^T u(q) \cdot \mathrm{d}q \tag{3.1}$$

subject to

$$\Sigma_p \delta_{ap}^k h_p^k = f_a^k \qquad \forall\ (a, k) \tag{3.2}$$

$$\Sigma_{p \in P_w} h_p^k = T_w^k \qquad \forall\ (w, k) \tag{3.3}$$

$$h_p^k \geq 0 \qquad \forall\ (p, k) \tag{3.4}$$

where $[u_w^k(T)] \equiv u(T)$ is the inverse of the system of demand functions $[T_w^k(u)] \equiv T(u)$, that is, $u(\cdot) = T^{-1}(\cdot)$. (The arc-chain formulation of the conservation eqns (3.2–3.4) can be easily replaced, if we wish, by the equivalent arc-node formulation.)

Lemma 1. The solution set of program P1 is identical to the solution set of program P2.

Proof: Integrating-by-parts the second term in the objective function of P2,

$$\begin{aligned} \int_0^T u(q)\, \mathrm{d}q &= [q \cdot u(q)]_0^T - \int_{q=0}^{q=T} q \cdot \mathrm{d}u(q) \\ &= T \cdot u(T) - \int_{q=0}^{q=T} q \cdot \mathrm{d}u(q) \end{aligned} \tag{4}$$

Substituting definitional eqns $[u(q) = y$ and $u(T) = u]$ and their inverses $[q = u^{-1}(y) = T(y)$ and $T = T(u)]$ in the right-hand side of (4) and in eqn (3.3) yields program P1. Thus any optimal solution of P2 is a feasible solution of P1 and yields the same value of the objective function as P2.

Conversely, integrating-by-parts the second term in the objective function of P1, and again making the above substitutions, yields program P2. ∎

To show (Lemma 3 below) that program P2 solves the equilibrium conditions (2.1–2.3) we reintroduce assumptions (i–vii) from P1, but do not assume that P2 satisfies the restrictive assumptions (viii) and (ix). Assumptions (i–iv) have the same role as in the case of program P1. Assumptions (v–vii) refer to the direct demand functions in P1, but it is easy to show that they imply analogous properties for the inverse demand functions in P2. For this we use the following well-known lemma.

Lemma 2. Definiteness of the Jacobian matrix $\mathbf{J}(T)$ of the demand functions, for all $u \geq 0$, is sufficient to ensure that the functions $T = T(u)$ are invertible for all $u \geq 0$.

Proof: This is, for example, a special case of Theorem 6 in Gale and Nikaidô (1959), which shows that (quasi) definiteness of the Jacobian matrix of a vector valued function over any convex region is sufficient to ensure that the functon is everywhere invertible in that region. (Note that the Jacobian matrix being everywhere nonsingular is not in itself sufficient to ensure that a function is everywhere invertible.) ■

Assumption (vii) together with Lemma 2 imply that the inverse functions $u = u(T)$ exist, hence $[\partial u(T)/\partial T] = [\partial T(u)/\partial u]^{-1}$, and assumptions (v–vii) imply that

(v′) $u_w^k = u_w^k(T)$ is a function of the full vector of O–D flows $\mathbf{T}$.
(vi′) $\mathbf{J}(u)$, the Jacobian matrix for the inverse demand functions, is symmetric.
(vii′) $-\mathrm{J}(u)$ is positive definite.

Assumption (vi′) ensures that the first derivative of the second term in (3.1) is $u(T)$, and assumptions (vii′) and (iv) ensure that (3.1) is convex.

It is now easy to show:

Lemma 3. Given assumptions (i–vii) above, the solution set of program P2 is identical to the solution set of the traffic equilibrium conditions (2.1–2.3).

Proof: Write out the Kuhn–Tucker (K–T) conditions for program P2. These contain variables T_w^k and inverse demand functions $u_w^k(T)$, but not the price variables u_w^k. Introduce definitional constraints $u_w^k = u_w^k(T)$. The inverse of these, i.e. $T_w^k = T_w^k(u)$, $\forall$ (w, k), exists (Lemma 2 above) and substituting $T_w^k = T_w^k(u)$ and $u_w^k = u_w^k(T)$ in the K–T conditions reduces these to the equilibrium conditions (2.1–2.3). To show that the K–T conditions, and hence (2.1–2.3) are equivalent to the solution set of P2, note that the constraints of P2 are linear and hence they satisfy a constraint qualification, hence the K–T condition necessarily hold at any optimum of P2. Further, the objective function of P2 is strictly convex and the constraint set is convex (since all constraints are linear), hence the K–T conditions are sufficient to ensure an optimum of P2. The K–T conditions are thus necessary and sufficient for a solution of P2 and are hence equivalent to the solution set of P2. ■

Combining Lemmas 1 and 3 we have:

Lemma 4. Given assumptions (i–vii) above, the solution set of program P1 is identical to the solution set of the traffic equilibrium conditions (2.1–2.3). ■

Demand functions are sometimes estimated in "inverse' form, that is, with prices stated as functions of quantities, $u = u(T)$. If such demand functions are not invertible then the direct demand function $T = T(u)$ do not exist. This does not invalidate the formulation P2, since P2 can be justified without reference to P1: in the user-equilibrium case program P2 solves the equilibrium problem (2.1–2.3), and in the system optimum case P2 has a direct optimization interpretation.

4. A ROLE FOR THE QUANTITY-PRICE MODEL

Despite the above discussion, there is a potential (limited) role for the quantity-price model, for cases in which the demand functions are not invertible or in which the explicit functional form of the inverse does not exist, or is unknown, or is difficult to compute. The quantity-price model P1 uses only the direct demand functions [in (1.1) and (1.3)] whereas the quantity-quantity model P2 uses the inverse demand functions [in (3.1)]. Thus, if direct demand functions are available and the inverse demand functions are not available, then model P1 can be used but model P2 can not. However, as we will see, even if the inverse demand functions are not available it may still be possible to use model P2.

Demand systems for which an inverse is not available are commonly used, and arise in two ways. First, any functional relationship among the elements of a vector valued function renders the function noninvertible [see a calculus text, e.g. Apostol (1969)]. This implies that some of the most popular travel demand functions, including most market-share type demand functions, gravity models and logit models are not invertible, since these have the property that the total number of trips into, or out of, some or all zones is fixed for all values of the travel price or cost vector $\mathbf{u}$, i.e. $\Sigma_{(w,k)\in S} T_w^k = \Sigma_{(w,k)\in S} T_w^k(u) =$ a constant, where S is a set of some or all origin-destination pairs and user types.

A second reason for the unavailability of inverse functions is that even when the inverse exists it may be difficult or impossible to state the explicit form of the inverse: for example, the scalar function $x = x(u) = a - bu - c \ln u$, $u > 0$, is everywhere invertible but there is no analytic function to represent the inverse. Of course, it is not always essential to have the *explicit* form of the inverse functions in order to solve problem P2, since gradient search algorithms require only that the functions be evaluated at particular *points*. Specifically, when solving problem P2, the prices $u(T)$ are required for particular values of demand T, and these can be computed from the uninverted functions $T = T(u)$, by using say Newton's method [Ortega and Rheinboldt (1970)]: note that this is easiest when the demand functions are separable, but even in that case it adds significantly to the cost of solving P2. Some optimization algorithms also require values of the derivatives $\partial u(T)/\partial T$ for given values of T, but these can be obtained by computing u from T as above, then computing $\partial T(u)/\partial u$ for this value of u, and solving $\partial u(T)/\partial T = [\partial T(u)/\partial u]^{-1}$. Again this is easiest if the demand functions are separable, in which case $\partial u_w^k(T_w^k)/\partial T_w^k = 1/(\partial T_w^k(u_w^k)/\partial u_w^k)$ for all (k, w). If the demand functions are not separable then computing inverse function values can be very costly in the absence of an explicit inverse, since it involves computing the solution of a system of nonlinear equations.

As a final (minor) comment on program P1 above, note that we have let the lower limit of integration on the second line integral in (1.1) be $u(0) = u(T = 0)$, rather than use the lower limit $u = 0$ used by Fernandez, Friesz and Aashtiani. This adds a constant to the objective function but does not otherwise affect the solution. However, this constant is important, if the value of the objective function is to be used as (the negative of) a meaningful measure of net benefit, as it can be used when P1 is set up to model a system optimum rather than user equilibrium. It is easy to check that using $u(0)$ rather than $u = 0$ yields the appropriate measure of consumer surplus, by sketching the usual demand and social marginal cost functions for the single O–D single link case, as in say Gartner (1980).

5. A PRICE-QUANTITY OPTIMIZATION MODEL

Finally, it is worth noting that, by analogy with the "quantity-price" model P1, a "price-quantity" model can be constructed, using the inverse of the travel time/cost or impedence functions and the direct demand function. Thus,

P3: Minimize

$$c \cdot f(c) - \int_{c(0)}^{c} f(\bar{c}) \cdot d\bar{c} - \int_{0}^{T} u(q) \cdot dq \tag{5.1}$$

subject to

$$\Sigma_p \delta_{ap}^k h_p^k = f_a^k(c) \qquad \forall\ (a, k) \tag{5.2}$$

$$\Sigma_{p \epsilon P_w} h_p^k = T_w^k \qquad \forall\ (w, k) \tag{5.3}$$

$$h_p^k \geq 0 \qquad \forall\ (p, k) \tag{5.4}$$

where $[f_a^k(c) \equiv f(c)$ is the inverse of the system of trip time/cost or impedance functions, $[c_a^k(f)] \equiv c(f)$, i.e. $f(\cdot) = c^{-1}(\cdot)$. Note that explicit nonnegativity constraints on f_a^k and T_w^k are not needed, since substituting (5.4) in (5.2) and (5.3) ensures $f_a^k(c) \geq 0$ and $T_w^k \geq 0$. Also, $f_a^k(c) \geq 0$ ensures $c_a^k \geq 0$, for all meaningful travel cost functions.

Analogous to Lemma 1 above we have:

Lemma 5. The solution set of program P3 is identical to the solution set of program P2.

Proof: Similar to the proof of Lemma 1 above, except that we integrate-by-parts the first, rather than the second, line integral term in the objective functions of programs P3 and P2. ■

Combining Lemmas 3 and 5 we obtain, analogous to Lemma 4:

Lemma 6. Given assumptions (i–vii) above, the soution set of program P3 is identical to the solution set of the traffic equilibrium conditions (2.1–2.3). ■

The discussion of the quantity-price model P1, in Sections 2 and 4 above, applies also to the price-quantity model P3. Thus, the nonlinear equality constraints (5.2) render the constraint set of P3 nonconvex, just as (1.3) rendered P1 nonconvex. In view of this and in view of Lemmas 3, 4 and 5 above, it is generally preferable to use the well-behaved (convex) model P2 instead of P3. However, with appropriate changes of wording, the discussion of the quantity-price model P1 in Section 4 above also applies to the price-quantity model P3. Thus, throughout Section 4 we can substitute the travel time/cost or impedance functions $f(v)$ for the inverse demand functions $T^{-1}(q)$. In particular, we conclude that if the functions $v = f^{-1}(\cdot)$ are available and if the functions $c = f(\cdot)$ are not available, then model P3 may be useful.

6. SOLVING THE QUANTITY-PRICE AND PRICE-QUANTITY MODELS

We discuss here solving only the quantity-price model P1, since methods for solving the price-quantity model P3 are closely analogous. Also, some relevant properties of program P1 have already been noted in Section 2 above. There are a number of approaches which can be adopted to solve program P1. For example, we could replace the nonlinear expressions in P1 with piecewise linear approximations and solve as a standard linear programming problem. However, this is not an efficient method since it does not take advantage of the special network structure of the problem.

A popular method for solving the quantity-quantity program P2 is the Frank–Wolfe feasible directions method [Leblanc *et al.* (1975)], hence it is particularly interesting to apply an analogous method to program P1. The main difference involved here is that there is a nonlinear expression $T_w^k(u)$, rather than a variable or constant T_w^k, on the right-hand side of (1.3). However, it turns out that the computations involved in applying the method to P1 or to P2 are almost identical, and indeed *are* identical if both and direct and inverse demand functions [$T = T(u)$ and $u = u(T)$] are available.

In applying the Frank–Wolfe algorithm to program P2 the direction finding subproblem at each iteration involves solving

P2′: Minimize

$$\nabla z(f, T) \cdot [f, T] = c(\bar{f}) \cdot f - u(\bar{T}) \cdot T$$

subject to (3.2–3.4)

where ∇ denotes the gradient (column) vector, $\nabla z(f, T) = [\partial z/\partial f, \partial z/\partial T]$.

In adapting the Frank–Wolfe method to program P1 we replace the nonlinear term $T_w^k(u)$ in (1.3) with a linear tangential approximation at each iteration, so that the direction finding subproblem to be solved at each iteration is,

P1′: Minimize

$$\nabla z(f, u) \cdot [f, u] = i(\bar{f})^T \cdot f - (\bar{u} \cdot \nabla T(\bar{u})^T) \cdot u$$

subject to (1.2), (1.4), with (1.3) replaced by,

$$\Sigma_{p \in P_w} h_p^k = \nabla T_w^k (\bar{u})^T \cdot u \tag{1.3″}$$

and the right hand side of (1.3″) is constrained to be nonnegative.

Now define variables

$$\tilde{T}_w^k = \nabla T_w^k(\bar{u}) \cdot u, \text{ and hence the matrix equation } \tilde{T} = \nabla T(\bar{u})^T \cdot u \tag{6}$$

Substituting this in program P1′, in the objective function and in the right-hand side of (1.3″), makes program P1′ formally identical to program P2′. The eqns (6) are not needed in the program since they merely define the u variables, which do not now appear elsewhere in the program. If we wish to solve (6) for u we can do so *after* solving P2′. Thus solving P2′ becomes identical to solving P1′.

One other step in the algorithm deserves note, and has already been remarked on in Section 5 above. This is the step in which the new gradient of the objective function of P2′ is computed. This requires computing $\bar{u}$ given $\bar{T}$, which may be easy if the inverse travel demand functions $u = u(T)$ are available, but if they are not, then we might use a linear approximation given by the inverse of (6), i.e. $u_{new} = \bar{u}_{old} + [\nabla T(u)^T]^{-1}\nabla T$. If the number of origin-destination pairs in the program is large, so that the matrix $\nabla T(u)$ to be inverted is large, this inversion can be a very costly step to be performed at each iteration. To reduce this cost, some method of updating the inverse at each iteration may be used, rather than reinverting from scratch at each iteration.

The final approach to solving P1 which we will mention here is Benders type decomposition [Benders (1962)]. This involves solving P1 by iterating between the following two steps until some convergence test is satisfied.

(a) hold the T_w^k's fixed, thus reducing P1 to a transportation assignment problem with fixed demands, thus,

P4: Minimize

$$\int_0^f c(x) \cdot \mathrm{d}x$$

subject to (1.2), (1.4) and

$$\Sigma_{p\epsilon P_w} h_p^k = \bar{T}_w^k \qquad \forall\ (w, k) \tag{1.3'''}$$

where $\bar{T}_w^k$ *is the fixed value of* T_w^k at the current iteration.

(b) Use the dual of P4 to find the direction in which to vary the $\bar{T}_w^k$'s and return to step (a).

Most of the computation is involved in solving program P4 in (a) at each iteration, but this computation can be greatly reduced by using the solution of P4 from the previous iteration as the starting point for solving P4 at the next iteration.

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