

## Link Outflow Rate Computing under Continuous Dynamic Loads

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### Abstract

The aim of this paper is to seek efficient and effective numerical methods for computing link outflow rates under continuous dynamic loads. Three methods are proposed and their respective properties are investigated, theoretically and numerically. Furthermore, some implications are presented for dynamic traffic assignment problem solving.

*Keywords:* Continuous dynamic loads, Dynamic traffic assignment (DTA), Link outflow (or departure) rate computing, The first-in-first-out (FIFO) principle

### Introduction

In the past decades there is an increasing number of papers which contribute to formulating the dynamic traffic assignment (DTA) problem. The latest review of this problem is given by Ben-Akiva *et al.* (2001), Boyce, Lee and Ran (2001), Carey (2001a), Friesz *et al.* (2001), Mahmassani (2001) and Peeta and Ziliaskopoulos (2001). However, so far no paper has been seen to specially report link outflow (or departure) rate computing under continuous dynamic loads. In fact, it is one of the most important problems in the solving of DTA. Usually, a DTA problem consists of four parts: flow propagation, travel choice, flow conservation and others (e.g., driver information systems, driver learning); among them the first two are the main parts of implementing a solution method of DTA. Link outflow rates as well as travel times define how network link flows change over time. Therefore, the performance of the method for computing link outflow rates directly affects the efficiency of solving DTA. The studies of ours on DTA have witnessed some interesting and rewarding results arising from the use of different methods to compute the link outflow rate.

The main purposes of this paper are (1) to report three methods for computing the link outflow rates given time-varying inflow (or arrival) rates we used in solving DTA, (2) to investigate their properties, theoretically and numerically, and (3) to provide some implications for the solving of DTA. This effort is hoped helpful for the understanding of transportation network dynamics and the solving of DTA on large sized networks.

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Without loss of generality, only consider one link in the network of interest in this paper. Also, the following assumptions are presumed:

- (i) Traffic enters only at the beginning of the link and exits only at its end;
- (ii) Traffic is not dissipated or created along the link, i.e., conserved along the link;
- (iii) Traffic exits in the same order as it enters, i.e., the first-in-first-out (FIFO) principle is followed.

The next section presents the three methods of the link outflow rate computing over time given the inflow rate and discusses their respective properties theoretically. Subsequently, some numerical experiments are carried out to further highlight the properties of these methods. The final section concludes this paper with some implications for the solving of the DTA problem.

### Link outflow rate computing

Consider one link in the network of interest and the time horizon is specified in  $[0, T)$ . The flow rate entering the link at time  $t \in [0, T)$  is assumed to be  $u(t)$ , which experiences travel time  $\tau(t)$  traversing the link, and traffic at time  $t$  exits from this link at the rate of  $v(t)$ . Since traffic is conserved along one link and follows the FIFO principle, traffic entering up to time  $t$  will exit up to time  $t + \tau(t)$ . Mathematically,

$$\int_0^t u(\omega) d\omega = \int_0^{t+\tau(t)} v(\omega) d\omega \quad (1)$$

Differentiating the above equation gives

$$v(t + \tau(t)) = u(t) / [1 + \tau'(t)] \quad (2)$$

Indeed, it is a shortcut to compute the link outflow rate by means of Eq. (2). In fact, simply replacing  $\tau'(t)$  with its finite difference approximation produces a progressive procedure for computing link outflow rates discretely. However, this is not a good way to generate link outflow rates. First, this would produce/accumulate large errors in the solving of DTA models. Secondly, computing the derivative of travel time function is relatively time consuming. Furthermore, fatally, this method does not always work. In fact, once  $\tau'(t) \leq -1$  (i.e., the FIFO violation turns up), this method fails. On the other hand, the FIFO rule is not the necessary condition to the DTA problem. Therefore, it is essential to seek efficient and effective methods for computing link outflow rates under continuous dynamic loads.

What follows is to construct intuitively the method of the outflow rate computing.

Divide  $[0, T)$  equally into  $N$  time intervals, indexed by  $i = 0, 1, \dots, N - 1$ , and then the length of each time interval is  $\delta = T/N$ . Usually, it is required that  $\delta$  is not greater than the free flow travel time traffic traverses the whole link. Let

$$\tau_i = \tau(i\delta), \quad u_i = \int_{i\delta}^{(i+1)\delta} u(\omega) d\omega, \quad \text{and} \quad v_i = \int_{i\delta}^{(i+1)\delta} v(\omega) d\omega. \quad (3)$$

In a discrete setting,  $\tau_i$  denotes the travel time which is determined by the traffic condition at the beginning of time interval  $i$ ; obviously,  $v_i/\delta$  produces the approximate outflow rate of interval  $i$ . When  $\delta \rightarrow 0$ , the series  $\{v_i/\delta\}_{i=0,1,\dots}$  approaches the link outflow rate  $v(t)$ .

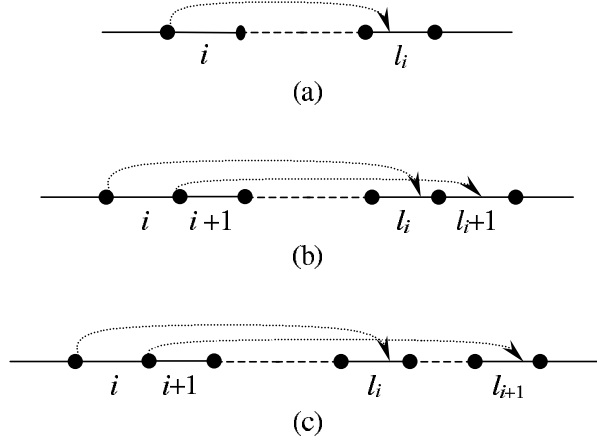


Fig. 1. Illustration of outflow distribution over time intervals.

#### Method 1

Intuitively, traffic entering the link in interval  $i$  should exit in interval  $l_i = i + \lceil \tau_i / \delta \rceil$ , as shown in Fig. 1. An easy way to understand this method is to suppose the inflow of one interval enters the link at the beginning instant of the interval and then, experiencing the same travel time, leaves the link collectively at one instant.

This method does not consider the FIFO rule. A simple way to embody the FIFO rule into this method is to add the following procedure to it. First define a function  $e(t) = t + \tau(t)$  which gives the instant the traffic entering the link at time  $t$  leaves the link; in the discrete setting,  $e_i = e(i\delta)$ . From now on the function  $e(t)$  (as well as  $e_i$ ) is referred to as an exit function of the link. Obviously, if  $e_i \leq e_{i+1}$  holds for all time intervals  $i$ , then the FIFO principle is kept.

Appealingly, this method significantly reduces the computing time relative to the Eq. (2) based method.

#### Method 2

Now let us look at Fig. 1(b). In the above method, the traffic entering the link in the same time interval is assumed to have the same travel time. However, it should be pointed out that their exit times may be different because of their different entry times. Consider two vehicles, one entering the link at the beginning of interval  $i$  and the other at the end of the interval. Then their exit times should be respectively  $e_i = i\delta + \tau_i$  and  $e_i + \delta = (i+1)\delta + \tau_i$ . This shows that the two vehicles exit the link in the different time intervals. The first vehicle entering the link at the beginning of interval  $i$  exits in interval  $l_i = i + \lceil \tau_i / \delta \rceil$  and then the last vehicle entering at the end of the interval should exit in interval  $l_i + 1$ . Specifically, if traffic distributes uniformly over intervals, the inflow in interval  $i$  should be split into the two intervals according to the following proportion

$$\begin{aligned}
 p_{l_i} &= \frac{(l_i + 1)\delta - e_i}{\delta} \\
 p_{l_{i+1}} &= \frac{e_i + \delta - (l_i + 1)\delta}{\delta} = \frac{e_i - l_i\delta}{\delta}
 \end{aligned}$$

Further, if the FIFO principle is considered, an additional procedure should be provided

to judge whether the FIFO is kept, similar to in Method 1. The FIFO principle means  $e_i \leq e_{i+1}$  for all  $i$ . If both  $e_i$  and  $e_{i+1}$  fall into the same interval, then

$$p_{l_i} = 1, \text{ and } p_{l_{i+1}} = 0;$$

furthermore, if  $e_i > e_{i+1} - \delta$ , then

$$\begin{aligned} p_{l_i} &= \frac{(l_i + 1)\delta - e_i}{e_{i+1} - e_i} \\ p_{l_{i+1}} &= \frac{e_{i+1} - (l_i + 1)\delta}{e_{i+1} - e_i} \end{aligned}$$

In terms of the computing time, this method is more efficient than the Eq. (2) based method.

### Method 3

When the link inflow rate and travel time are continuous in terms of time  $t$ , the two vehicles should have different travel time even though they enter the link successively. This implies that in the discrete setting the assumption that the vehicles entering the link in the same time interval have the same travel time may be refined further.

Consider two vehicles enter the link respectively at the beginnings of intervals  $i$  and  $i + 1$  and their travel times are  $\tau_i$  and  $\tau_{i+1}$ , respectively. Hence, they exit the link at times

$$e_i = i\delta + \tau_i, \tag{4}$$

$$e_{i+1} = (i + 1)\delta + \tau_{i+1}, \tag{5}$$

respectively. It is more reasonable to assume that the inflow  $u_i$  exits and spreads over the time span from  $e_i$  to  $e_{i+1}$ . Here the FIFO principle is postulated and hence, using the previous notation, there are only two cases:  $l_i = l_{i+1}$  and  $l_i < l_{i+1}$ .

If  $l_i = l_{i+1}$  holds, then all inflow in interval  $i$  exits in a single interval (i.e., interval  $l_i$ ).

If  $l_i < l_{i+1}$  and the inflow in interval  $i$  exits uniformly over  $[e_i, e_{i+1})$ , then the fraction of the inflow in interval  $i$  that exits in interval  $l_i$  is

$$p_{l_i} = \frac{(l_i + 1)\delta - e_i}{e_{i+1} - e_i};$$

another fraction of the inflow in interval  $i$  leaving in interval  $l_{i+1}$  is

$$p_{l_{i+1}} = \frac{e_{i+1} - l_{i+1}\delta}{e_{i+1} - e_i};$$

further, if  $l_i + 1 < l_{i+1}$ , the remaining fraction of the inflows are equally assigned to each time interval  $j$  from  $l_i + 1$  to  $l_{i+1} - 1$ , i.e.,

$$p_j = \frac{\delta}{e_{i+1} - e_i}, \quad l_i + 1 \leq j \leq l_{i+1} - 1.$$

In the above discussion, the traffic is assumed to distribute uniformly in each time interval. In fact, we may have a more general conclusion, which can be stated as

**Theorem 1** Under the assumptions (i)–(iii) presumed in the Introduction, both

$$u_i = \int_{e_i}^{e_{i+1}} v(\omega) d\omega \quad (6)$$

and

$$v_i = \int_{e_i^{-1}}^{e_{i+1}^{-1}} u(\omega) d\omega \quad (7)$$

hold in the discrete setting, where  $e^{-1}(\cdot)$  denotes the inverse of the exit function  $e(\cdot)$ .

It can be shown the above theorem from Eq. (1) and its proof is left to readers.

Formula (6) shows we can finely spread the inflow in interval  $i$  over the time span  $[e_i, e_{i+1})$ , based on the shape of the inflow rate function. Formula (7) shows that once the inverse of the exit function is known, the link outflow rate can be generated directly. The two formulae above provide the two ways to refine the link outflow rate computing; however, this kind of refinement would effect more computing time in the solving of DTA. However, if in a discrete setting the length of time interval is relatively large, this kind of refinement would be essential and rewarding. Formulae (6) and (7) should also find their niches in the studies of dynamic bottleneck modeling.

If  $l_i = l_{i+1}$  or  $l_i = l_{i+1} - 1$  holds for all  $i$ , then the computing time required by Method 3 is the same as that required by Method 2. However, this is not always true and hence the computing time required by Method 3 is generally more than that required by the other two methods, especially when the inflow rate often changes drastically. Though, the expected computing time of this method could be less than the Eq. (2) based method.

The three methods described above all have the mechanism which checks if the FIFO principle is violated; however, they can easily be revised to accommodate the situation where the FIFO violation turns up in the solving of DTA. Especially, the first two methods can be applied directly to the situation with the FIFO violation. In the time dependent analysis of traffic, it is essential to consider the impacts of queueing under the concepts of point or physical queue. The methods introduced here can be applied with some modification to scenario of this kind.

## Numerical experiments

This section considers the impacts of the use of different computational methods on the link outflow rate obtained. Recently, Carey et al. (2000) presents a travel time model, specially designed for DTA modeling, as follows

$$\tau(t) = f(\beta u(t) + (1 - \beta)v(t + \tau(t))), \quad 0 \leq t < T$$

where  $\beta \in [0, 1)$  is a parameter to be calibrated, balancing the effects of the inflow rate at time of entry and the outflow rate at the exit time. The solution method of this model is given in Carey (2001b). Specifically, a linear form of this model is adopted in this paper as

$$\tau(t) = \tau_0 + \alpha[\beta u(t) + (1 - \beta)v(t + \tau(t))] \quad (8)$$

where  $\tau_0$  denotes the free-flow travel time, and  $\alpha$  is a parameter to be calibrated. In this paper, assume that  $\tau_0 = 20.0$ ,  $\alpha = 0.5$  and  $\beta = 0.5$ .

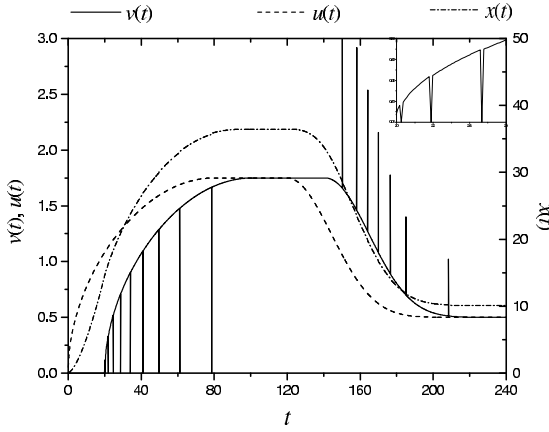


Fig. 2. Solutions from Method 1.

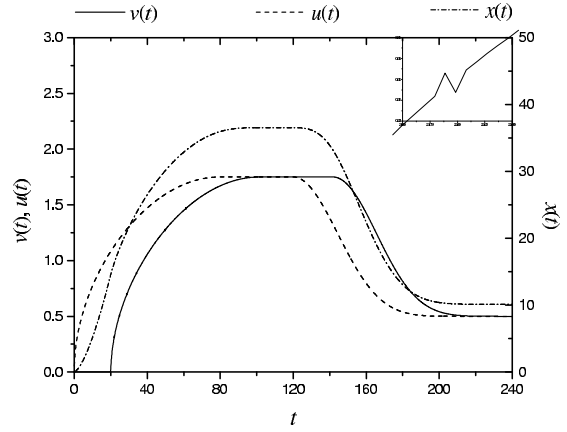


Fig. 3. Solutions from Method 2.

In addition, assume that  $T = 240$ , and that the time-varying demand is given by

$$q(t) = \begin{cases} \frac{7}{4} \sqrt{\sin\left(\frac{\pi t}{160}\right)} & 0 \leq t < 80 \\ \frac{7}{4} & 80 \leq t < 120 \\ \frac{1}{2} + \frac{5}{4} \sin^{10}\left(\frac{\pi t}{240}\right) & 120 \leq t < T \end{cases}$$

Here only one link is considered and hence  $q(t)$  defines the inflow rate of the link, i.e.,  $u(t) = q(t)$ .

Figs. 2–5 illustrates the characteristics of link outflow rates generated by the different computational methods, including the Eq. (2) based method. Now let us observe them in order. It can be imagined that when the travel time is constant, the outflow rate curve is the very shift of the inflow rate curve. Therefore, under the constant link travel time, the three methods proposed here as well as the Eq. (2) based method produce the same outflow rate curve. Generally, the travel time is a function of the inflow and outflow rates, implicitly or explicitly. The interactions of traffic inherent in functions of this kind make the travel time traversing the link larger than the constant free flow travel time. Therefore, the outflow is spread over a wider time range when the inflow increases. This point can be seen from Fig. 2, as well as from the other figures later. Method 1 directly maps the inflow in one interval into the outflow in another interval. If the difference in travel times of vehicles entering the link between two successive intervals, say,  $i$  and  $i + 1$ , is greater than the length of time interval then no traffic would exit in interval  $j$  ( $l_i < j < l_{i+1}$ ); under the FIFO assumption the outflow rate between the two exit intervals equals zero. The analysis is confirmed in Fig. 2, where the outflow rate generated by Method 1 is zero in some intervals and equals the corresponding inflow rate otherwise. In addition, when the inflows decrease, the outflow rate of one interval might be greater than the inflow rate if the inflows in two successive time intervals exit the link in the same interval, as shown in Fig. 2. The existence of this kind of outliers seems to make it difficult to adopt the method for computing the link outflow rates under continuous dynamic loads.

In Fig. 3, this situation is improved significantly by introducing Method 2. Splitting the inflow in one interval into two successive intervals can be expected to enable the fewer number of outliers on the outflow rate curve when the flow increases or decreases. This

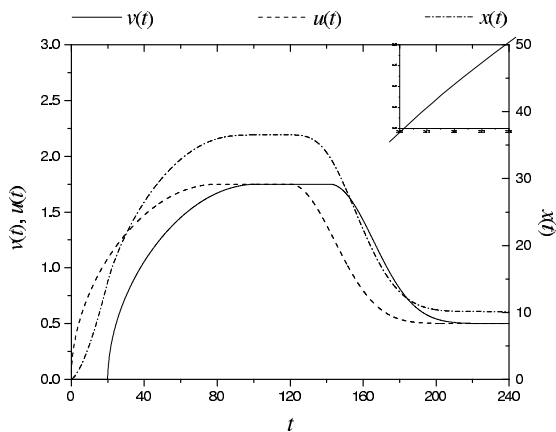


Fig. 4. Solutions from Method 3.

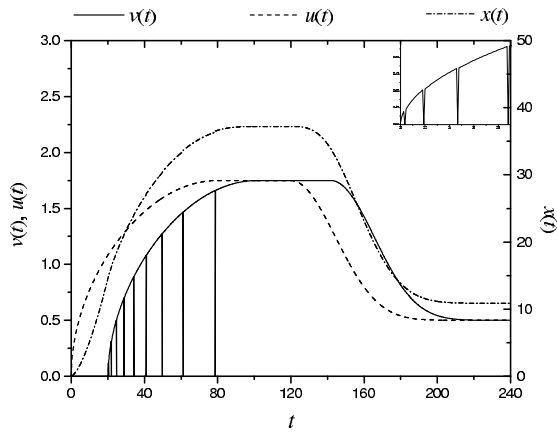


Fig. 5. Solutions based on Eq. (2).

point is proved in Fig. 3, where there are no significant outliers. However, on the outflow rate curve, there are some visible spikes which make the curve “non-smooth”.

The result in Fig. 4 is improved obviously because of the use of Method 3, where the outflow rate curve seen is “smooth”. Obviously, the outflow rate curve in this figure can better represent how network link flows change over time. This implies that Method 3 can fully capture time-varying outflows and their properties under continuous dynamic loads.

Interestingly, the curve of the number of vehicles (i.e.,  $x(t)$ ) on the link seems to be independent of the methods of the outflow rate computing (refer to Figs. 2–4). This may be substantiated by observing the travel time curve, which also is independent of the used methods in this example.

Now let us observe the result generated by Eq. (2). Fig. 5 further confirms the previous criticism that the error accumulation effects the undesired link outflow rates and number of vehicles on the link; the latter can be seen in this figure where the number of vehicles on the link are larger than that produced by the methods proposed in this paper. Furthermore, the zero-outflow-rate outliers seen in Fig. 2 come up in this figure too.

The phenomenon presented above is not dependent on model (8). The further experiment on the likes of  $\tau(t) = f(x(t))$  substantiates this claim. The independence of this phenomenon from the inflow profile given as well as the length of time interval has also been observed in our other experiments carried out.

In summary, the proposed methods here have advantages over the Eq. (2) based method. Specifically, the first two methods are better than the Eq. (2) based method in terms of computing time and the last two are better than it in terms of computational precision.

Furthermore, the approximate independence of the number of vehicles on the link and the travel time from among the newly proposed solution methods enables the use of the first two ones in the solving of DTA on large sized networks if the computing time required is more important than the computation precision in outflow rates. However, if the analysis to be carried out is based on small networks the last two methods, especially the third one, are recommended.

## Conclusions

Three methods for computing the link outflow rates under continuous dynamic loads have been presented with a theoretic and numerical analysis of their respective properties. The methods proposed here are developed presuming the FIFO principle and however can be extended to accommodate the situation where the FIFO violation comes up.

Compared to the existing method based on Eq. (2), the newly proposed methods are more efficient and more effective, and are of great benefit to implementing DTA on a large sized transportation network. Although the first two methods are not fully good when applied to link outflow rate computing on a small network, they are expected to be more efficient than the last one in terms of the computing time. When the analysis of network dynamics to be carried out is on a small network, the third method is strongly recommended in case the potential analytical properties are lost.

The conclusion above is based on numerical results obtained on a one-link network. Therefore, further experiments should be carried out so that the conclusion stated here can be substantiated and the more general conclusion can be drawn.

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