IMACC 2017

December 12–14, 2017

Lattice Reductions over Euclidean Rings with Applications to Cryptanalysis

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Backgrounds

Lattice-based cryptography

- post-quantum cryptography
- fully homomorphic encryption
- many other applications

Lattices (classical)

- M: a free $\mathbb{Z}$-module, i.e. closed under addition/multiplication by $\mathbb{Z}$
- A free $\mathbb{Z}$-module has a $\mathbb{Z}$-basis $(b_1, \ldots, b_n) \subset M^n$ s.t.

$$M = \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}b_n$$

- M has "infinitely" many number of basis
- n, the size of the basis, is "invariant" under the choice of the basis.
- n: the rank (or dimension) of M, written as $n := \mathrm{rk}(M)$

Backgrounds

Lattice reduction

- Some basis of lattices are good, but some are not.
- Many lattice problems (SVP/CVP) are easier to solve, if a "good" basis is given.
- Informally, a good basis consists of "reasonably small" and "almost orthogonal" components.
- "Lattice reduction" is to find such a "good" basis from an arbitrary basis.
- LLL-algorithm is one of the most popular algorithms for lattice reduction.
- A key tool not only for lattice-based crypto, but also for various cryptanalysis (e.g. RSA attack)

LLL-algorithm

LLL-algorithm (classical)

- Given a $\mathbb{Z}$-basis of a lattice M, find an LLL-reduced basis.
- (Gram-Schmidt orthogonalization) Given a basis $(b_1, \ldots, b_n) \subset \mathbb{Q}^n$, $(b_1^*, \ldots, b_n^*) \subset \mathbb{R}^m$ is GS orthogonalization, if $b_1^* = b_1$ and

$$b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{i,j} b_j^* \text{ for } \mu_{i,j} = \langle b_i, b_j^* \rangle / \langle b_j^*, b_j^* \rangle.$$

- (LLL-reduced basis) A basis $(b_1, \ldots, b_n)$ is LLL-reduced w.r.t $\delta > 0$ if

$$|\mu_{i,j}| \leq 1/2 \text{ for } 1 \leq j < i \leq n \text{ (size reduced)} ,$$

$$||b_i^*||^2 \geq (\delta - \mu_{i,i-1}^2)||b_{i-1}^*|| \text{ for } 1 < i \leq n \text{ (Lovasz conditions)}$$

Note

- LLL algorithm outputs a small vector of lattices
  - If $(b_1, \ldots, b_n)$ is LLL-reduced w.r.t. $\delta = 3/4$, then $||b_1|| \leq 2^{(n-1)/4} \det(M)^{1/n}$.
- Its running time depends on the size of the dimension, n.

Notations

Number fields

- $h \in \mathbb{Z}[t]$: an irreducible polynomial of degree n.
- $L := \mathbb{Q}[t]/h(t)$: a number field of degree n.
- $\mathbb{Z}_L$: a ring of integers of L (e.g. $\mathbb{Z}_{\mathbb{Q}} = \mathbb{Z}$)

Ideal lattices

- An ideal $\mathcal{I} \subset L$ is a $\mathbb{Z}_L$-submodule of L, thus trivially a $\mathbb{Z}$-module.
- $\mathcal{I}$ is free, since $\mathbb{Z}$ is a PID (Principal Ideal Domain).
- Thus, it is a $\mathbb{Z}$-lattice and called as "ideal-lattice".
- Typically, $\mathrm{rk}(\mathcal{I}) = n = [L : \mathbb{Q}]$.

Motivations

In cryptography,

- L typically has a proper subfield $K \neq \mathbb{Q}$.
- E.g. ℓ-th cyclotomic field $L = \mathbb{Q}[t]/(t^\ell + 1)$ has a subfield $K = \mathbb{Q}[t]/(t^k + 1)$, where $\ell = 2^l$ for some $l > 0$ and $k \mid \ell$.

Ideal lattice as a $\mathbb{Z}_K$-module

- An ideal $\mathcal{I} \subset L$ is also a $\mathbb{Z}_K$-module for a subfield $K \subset L$.
- If $\mathbb{Z}_K$ is a PID, then $\mathcal{I}$ is a free $\mathbb{Z}_K$-module ($\mathbb{Z}_K$-lattice), thus

$$\mathcal{I} = \mathbb{Z}_K\beta_1 \oplus \cdots \oplus \mathbb{Z}_K\beta_d$$

for a basis $(\beta_1, \ldots, \beta_d) \in L^d$ and $d = [L : K]$.

Matrices of Lattices

- We consider a $\mathbb{Z}_K$-lattice as a $d \times d$ matrix $M_\mathcal{I} = [\beta_1 | \ldots | \beta_d] \in K^{d\times d}$, instead of $[b_1 | \ldots | b_n] \in \mathbb{Q}^{n\times n}$, where $n = [L : \mathbb{Q}]$ and $d = [L : K]$.

Our goal

Motivation

- If a $\mathbb{Z}$-lattice $M \subset L$ can also be considered as a free $\mathbb{Z}_K$-module, $rk_{\mathbb{Z}_K}(M) = [L : K] = d$ is smaller than $rk_{\mathbb{Z}}(M) = [L : \mathbb{Q}] = n$.
- Smaller dimension, faster LLL-algorithm?

LLL algorithm over $\mathbb{Z}_K$

- We restrict our concern to "norm-Euclidean domain" $\mathbb{Z}_K$ (that is, Euclidean domain w.r.t. algebraic norm $N_{K/\mathbb{Q}}$).
- We propose two heuristic LLL algorithms running over $\mathbb{Z}_K$-lattices.

Technical hurdles

- For "GS orthogonalization", one needs to define "inner product" over $K^d \times K^d$.
- What would be analogous notions for "size reduced" and "Lovasz conditions"?

Related works

Related works

- (Napias '96) over Gaussian integers, more generally, quadratic norm-Euclidean domain
  - use "Hermitian product";
  - Euclidean norm (induced by Hermitian product) and algebraic norm coincides, i.e. $N_{K/\mathbb{Q}}(a + b\iota) = a^2 + b^2 = ||(a, b)||^2$.
- (Fieker-Phost '96) over arbitrary Dedekind domain, using pseudo-basis
  - inner product induced by Hermitian product;
  - in a size-reduction step, given $a \in K$, tried to find $q \in \mathbb{Z}_K$ s.t. $Tr_{K/\mathbb{Q}}((a - q)\overline{(a - q)})$ is minimal (in general, not easy to do so).
- (Gan-Ling-Mow '09) over complex fields
  - basically the same as Napias's;
  - does not consider the number field structures.
- (Fieker-Stehlé '10) over arbitrary Dedekind domain, using pseudo-basis
  - convert $\mathbb{Z}_K$-lattice into $\mathbb{Z}$-lattice of a higher dimension;
  - LLL-algorithm is carried over $\mathbb{Z}$-lattice.

Mathematical Background

Euclidean domain

- A ring R is Euclidean if $\exists \phi : R \to \mathbb{N}$ s.t. $\phi(a) \leq \phi(ab)$ for $0 \neq a, b \in R$ and there exists q and $r \in R$ s.t.

$$a = bq + r \text{ with } r = 0 \text{ or } \phi(r) < \phi(b).$$

- E.g. $\mathbb{Z}$ with $\phi(a) = |a|$ and division algorithm

Norm-Euclidean domain

- If $\mathbb{Z}_K$ in K is Euclidean w.r.t. $\phi(a) = |N_{K/\mathbb{Q}}(a)|$, then $\mathbb{Z}_K$ is called norm-Euclidean.
- (Example 1.) $\mathbb{Z}_K$ for $K = \mathbb{Q}(\zeta_k)$, the k-th cyclotomic field, is norm-Euclidean iff

$$k \in \{1, 3, 4, 5, 7, 8, 9, 11, 12, 15, 16, 20, 24\}^a$$

- (Example 2.) $\mathbb{Z}_K$ for $K = \mathbb{Q}(\sqrt{-\alpha}, \sqrt{\beta})$ is norm-Euclidean iff

$$\begin{aligned} &\alpha = 1, \quad \beta = 2, 3, 5, 7; \\ &\alpha = 2, \quad \beta = -3, 5; \\ &\alpha = 3, \quad \beta = 2, 5, -7, -11, 17, -19; \\ &\alpha = 7, \quad \beta = 5.^b \end{aligned}$$

[a] [Lenstra '75, Masley '75, Ojala '79]

[b] [Lemmermeyer '11]

Mathematical Background

Euclidean minimum

- "norm-Euclideanity" leads us to consider Euclidean minimum of K, $\mathfrak{M}(K)$.
- For any $\xi \in K$, $\exists q \in \mathbb{Z}_K$ s.t. $|N_{K/\mathbb{Q}}(\xi - q)| < \mathfrak{M}(K) < 1$.
- E.g. $\mathfrak{M}(\mathbb{Q}) = 1/2$

Euclidean minimum of $K = \mathbb{Q}(\zeta_k)$

k	1	3	4	5	7	8	9	12	15	16	20	24
$\mathfrak{M}(\mathbb{Q}(\zeta_k))$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{4}$

Table: Euclidean minimum of k-th cyclotomic fields [Lezowski '14]

Euclidean minimum of $K = \mathbb{Q}(\sqrt{-\alpha}, \sqrt{\beta})$

(α, β)	(1,2)	(1,3)	(1,5)	(1,7)	(2,-3)	(2,5)	(3,2)
$\mathfrak{M}(\mathbb{Q}(\sqrt{-\alpha}, \sqrt{\beta}))$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{11}{16}$	$\geq \frac{1}{4}$
(α, β)	(3,5)	(3,-7)	(3,-11)	(3,17)	(3,-19)	(7,5)	
$\mathfrak{M}(\mathbb{Q}(\sqrt{-\alpha}, \sqrt{\beta}))$	$\frac{1}{4}$	$\frac{4}{9}$	≤ 0.46	$\frac{13}{16}$	< 0.95	$\frac{9}{16}$	

Table: Euclidean minimum of biquadratic fields [Lemmermeyer '11]

Size reduction over norm-Euclidean ring

Size reduction (classical)

- Size reduction is actually the same as "for any given $\mu \in \mathbb{Q}$, find $q \in \mathbb{Z}$ s.t. $|\mu - q| < 1/2$".
- This can be done by a simple rounding, i.e. $q = \lfloor \mu \rceil$.
- Note that $\mathfrak{M}(\mathbb{Q}) = 1/2$ and $|\cdot|$ is the Euclidean function for $\mathbb{Z}$.
- Recall, in size reduction step, b_i is set to be $b_i - q_j b_j$, where $q_j = \lfloor \mu_{i,j} \rceil$ for $\mu_{i,j} \in \mathbb{Q}$.

Rounding function in norm-Euclidean ring

- For size reduction in K, we need an algorithm, for any $\xi \in K$, to find $q \in \mathbb{Z}_K$ s.t. $|N_{K/\mathbb{Q}}(\xi - q)| \leq \mathfrak{M}(K)$
- (Rounding function) For $a \in K$, we write $a = \sum_{i=0}^{n-1} a_i \zeta^i$ where $a_i \in \mathbb{Q}$. Define

$$\lfloor a \rceil := \sum_i \lfloor a_i \rceil \zeta^i.$$

- However, $|N_{K/\mathbb{Q}}(a - \lfloor a \rceil)| < \mathfrak{M}(K)$ does not hold in general.

Rounding algorithm for norm-Euclidean ring

Algorithm 1 Rounding algorithm for norm-Euclidean rings

Input A norm-Euclidean number field K, its Euclidean minimum $\mathfrak{M}(K)$, the unit group $K^\times$ of K, and an element $a \in K$

Output $q \in \mathbb{Z}_K$ such that $N_{K/\mathbb{Q}}(a - q) \leq \mathfrak{M}(K)$

1: Compute $r := a - \lfloor a \rceil$
2: **if** $N_{K/\mathbb{Q}}(r) \leq \mathfrak{M}(K)$ **then**
3: **return** $q := \lfloor a \rceil$
4: **else**
5: **repeat**
6: $u \leftarrow_\$ K^\times$
7: **until** $N_{K/\mathbb{Q}}(ur - \lfloor ur \rceil) \leq \mathfrak{M}(K)$
8: **end if**
9: **return** $q := \lfloor a \rceil + u^{-1} \lfloor ur \rceil$

Notes

Notes on the rounding algorithm

- The algorithm may not terminate, but it is unlikely to happen.
- In our experiments, the unit u is chosen from a power of a fundamental unit, e.g. $u \leftarrow v^i$ for a fundamental unit v.
- In the case of $K = \mathbb{Q}(\zeta_{16})$,
  - For 97% of 200,000 uniformly chosen random elements, it suffices to run the simple rounding (i.e. $q = \lfloor a \rceil$ is the desired output).
  - For the rests, it was enough to work with only a few units of the form v^i for $1 \leq i \leq 3$, where $v = (\zeta_{16}^6 + \zeta_{16}^4 + \zeta_{16}^2)$ is a fundamental unit.

LLL-algorithm over norm-Euclidean rings

Hermitian product

- Define a blinear map $H : K^d \times K^d \to K$ by

$$(\mathbf{v}, \mathbf{w}) \mapsto \sum_{i=1}^{d} v_i \overline{w}_i,$$

  where $\bar{\cdot}$ denotes the complex conjugation.
- GS-orthogonalization is analogously defined w.r.t. H.

LLL-reduced condition

- A basis $(\beta_1, \ldots, \beta_d) \subset K^d$ is called $\mathbb{Z}_K$-LLL-reduced w.r.t $\delta > 0$ if
  - $N_{K/\mathbb{Q}}(\mu_{i,j}) \leq \mathfrak{M}(K)$ for $1 \leq j < i \leq n$ (size reduced)
  - $N_{K/\mathbb{Q}}\big(B_i + \mu_{i,i-1}\overline{\mu_{i,i-1}} B_{i-1}\big) \geq \delta \cdot N_{K/\mathbb{Q}}(B_{i-1})$ for $1 < i \leq n$ (Lovasz condition), where $B_i = H(\beta_i^*, \beta_i^*)$.

Cautions

- Unlike the classical case, the Lovasz condition cannot be replaced with $N_{K/\mathbb{Q}}(B_i) \geq \big(\delta - \mu_{i,i-1}\overline{\mu_{i,i-1}}\big) \cdot N_{K/\mathbb{Q}}(B_{i-1})$.
- This is because the triangle inequality does not hold w.r.t. $N_{K/\mathbb{Q}}$.

LLL-algorithm over norm-Euclidean rings

Input a basis $\{\mathbf{b}_1, \cdots, \mathbf{b}_d\} \subset K^d$, $\mathfrak{M}(K)$, the unit group $K^\times$, and $\delta > 0$.
Output LLL-reduced basis $\{\mathbf{b}_1, \cdots, \mathbf{b}_d\}$.

Compute the Gram-Schmidt basis $\{\mathbf{b}_1^*, \cdots, \mathbf{b}_d^*\}$ with respect to the bilinear map $H(\cdot, \cdot)$
Compute the coefficients $\mu_{i,j} = H(\mathbf{b}_i, \mathbf{b}_j^*)/H(\mathbf{b}_j^*, \mathbf{b}_j^*)$ for $1 \leq j < i \leq d$ and $B_i = H(\mathbf{b}_i^*, \mathbf{b}_i^*)$ for $1 \leq i \leq d$.
Set $k = 2$
while $k \leq d$ **do**
    for $j = k - 1$ **to** 1 **do**
        Compute $q_j \in \mathbb{Z}_K$ such that $N_{K/\mathbb{Q}}(\mu_{k,j} - q_j) \leq \mathfrak{M}(K)$ using Algorithm 1
        Set $\mathbf{b}_k = \mathbf{b}_k - q_j \cdot \mathbf{b}_j$
        Update $\mu_{k,j} = H(\mathbf{b}_k, \mathbf{b}_j^*)/H(\mathbf{b}_j^*, \mathbf{b}_j^*)$ and B_k for $1 \leq j \leq k$
    end for
    if $N_{K/\mathbb{Q}}\big(B_k + \mu_{k,k-1}\overline{\mu_{k,k-1}}B_{k-1}\big) \geq \delta \cdot N_{K/\mathbb{Q}}(B_{k-1})$ **then**
        $k = k + 1$
    else
        Swap $\mathbf{b}_k$ and $\mathbf{b}_{k-1}$
        Update $\mathbf{b}_k^*, \mathbf{b}_{k-1}^*, B_k, B_{k-1}$, and $\mu_{i,j}$ for $1 \leq i, j \leq s$
        $k = \min\{2, k - 1\}$
    end if
end while

Experimental Results

Lattices

- The lattice $\mathcal{L}$ is generated by rows of the matrix in $K^{d\times d}$,

$$\begin{pmatrix} q & 0 & \cdots & \cdots & 0 \\ \gamma_1 & 1 & \cdots & \cdots & 0 \\ \gamma_2 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{d-1} & 0 & \cdots & \cdots & 1 \end{pmatrix},$$

  where q and $\gamma_i \in \mathbb{Z}_K$.
- This shape of the lattice basis appears in several cryptanalysis.

Parameter choices

- We carried out the reduction of lattices over $K = \mathbb{Q}(\zeta_k)$ for $k = 5, 8, 16$ of dimension $10 \leq d \leq 50$.
- If the lattices are considered as $\mathbb{Z}$-lattices, the dimension 50 over $\mathbb{Z}_K$ corresponds to the dimension 200 over $\mathbb{Z}$ when $k = 5$ or 8, or 400 when $k = 16$.
- $\delta = 3/4$

Experimental Results

Output quality

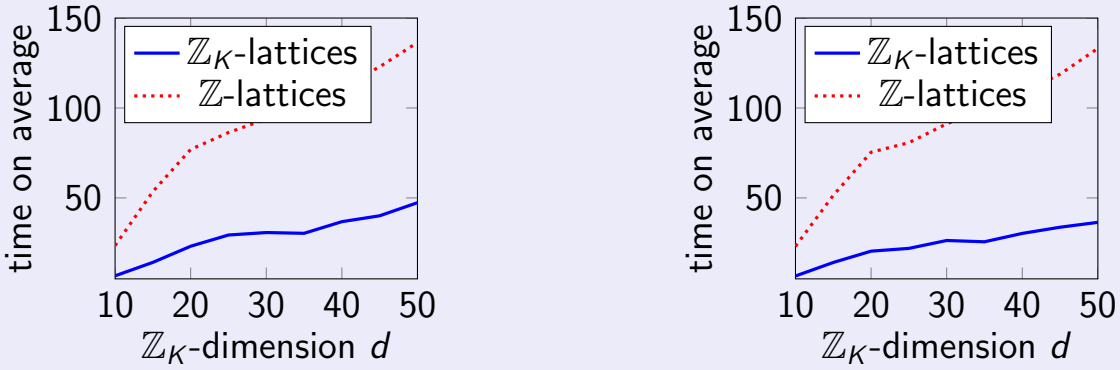
- Theoretically, we have no guarantee on the output quality.
- In biquadratic case, it is possible to guarantee output quality using a different type of inner product, but its practical performance is worse than general case that uses Hermitian product (see our paper).
- Let $n = d \cdot [K : \mathbb{Q}]$, a dimension of $\mathcal{L}$ over $\mathbb{Z}$
- For $\mathbf{v} := (v_1, \ldots, v_d) \in K^d$, define $||\mathbf{v}||_\infty = \max_i ||v_i||_\infty$.
- As a measure of output quality, we use

$$C := \frac{||\mathbf{b}_1||_\infty}{N_{K/\mathbb{Q}}(\det(\mathcal{L}))^{1/n}},$$

  where the volume of $\mathcal{L}$ over $\mathbb{Z}$ is the same as $N_{K/\mathbb{Q}}(\det(\mathcal{L})) = N_{K/\mathbb{Q}}(q)$.
- The classical Hermite factor is defined with $||\cdot||_2$ norm.
- Heuristically, we observe that $C \approx 1.02^n$.
- Taking $||\mathbf{v}||_2 \leq \sqrt{n}||\mathbf{v}||_\infty$ into account, the Hermite constant of our reduction is $\lesssim 1.02 n^{1/2n}$.
- As n grows, Hermite constant becomes close to the average Hermite factor 1.02.

Experimental Results

Timing results



(a) $\mathbb{Z}_K = \mathbb{Z}[\zeta_5]$

(b) $\mathbb{Z}_K = \mathbb{Z}[\zeta_8]$

Figure: Time comparison of running time

Note

- Our $\mathbb{Z}_K$-lattice reduction is about 3 times faster than the classical reduction done over $\mathbb{Z}$.
- We did not attempt to compare our naive implementation with already well-optimized LLL implementation.
- For the consistency of the comparison, we used our own implementation for both $\mathbb{Z}$ and $\mathbb{Z}_K$-reduction (see our SAGE codes in Appendix).

Applications

Sieving in exTNFS (by K.-Barbulescu)

- exTNFS is a best known algorithm to solve the DLP over $\mathbb{F}_{p^n}$ (n: composite, p: not small).
- In a step called special-q method in exTNFS method, we need to consider: find a small basis of the lattice

$$M_{\mathfrak{Q}} := \left\{ (a_0, \ldots, a_{\tau-1}) \in \mathbb{Z}[\iota]^{\tau} : \left(\sum_{i=0}^{\tau-1} a_i \alpha_f \right) \equiv 0 \mod \mathfrak{Q} \right\},$$

  where $\mathfrak{Q}$ is a prime ideal in $\mathbb{Z}_L$ for $L = K[\alpha] = K[x]/f(x)$ and $K = \mathbb{Q}[\iota] = \mathbb{Q}[t]/h(t)$.
- $M_{\mathfrak{Q}}$ is the τ-dimensional $\mathbb{Z}_K$-lattice.
- A classical approach is to consider the lattice as $\mathbb{Z}$-lattice.

Parameters for BN curves

- Set $h(t) = \Phi_5(t) = t^4 + t^3 + t^2 + t + 1$ so that $K = \mathbb{Q}(\zeta_5)$ and $\mathbb{Z}_K = \mathbb{Z}[\zeta_5]$.
- Set $f(x) = x^3 - x^2 - u$, where $p = P(u)$ and $P(x) = 36x^4 + 36x^3 + 24x^2 + 6x + 1$ and $u = 2^{158} - 2^{128} - 2^{68} + 1$
- Set $L = K[x]/f(x)$.

Special-$\mathfrak{q}$ in BN curve parameters

Example (cont.)

- Take a prime ideal $\mathfrak{Q} = \langle \mathfrak{q}, \alpha_f - \gamma \rangle \subset L$ and $\tau = 2$.
- where,

$$\mathfrak{q} = (q) = \big(-461479\zeta_5^3 - 383970\zeta_5^2 - 265505\zeta_5 - 303923 \big);$$

$$\gamma = 169465786435052577 63313.$$

- Then $M_{\mathfrak{Q}} = \begin{pmatrix} q & 0 \\ -\gamma & 1 \end{pmatrix}$.
- We obtained

$$\mathrm{LLL}\,(M_{\mathfrak{Q}}) = \begin{pmatrix} 532\zeta_5^3 + 850\zeta_5^2 + 179\zeta_5 - 464 & 224\zeta_5^3 + 132\zeta_5^2 - 13\zeta_5 + 367 \\ -649\zeta_5^3 + 186\zeta_5^2 + 661\zeta_5 + 73 & 11\zeta_5^3 - 264\zeta_5^2 + 35\zeta_5 - 71 \end{pmatrix}.$$

- Note that $\log_2(850) \approx 9.73$ and $\log_2\left(N_{K/\mathbb{Q}}(q)^{\frac{1}{\tau m}}\right) \approx 9.29$.

Future works

Open questions

- Apply to cryptanalysis of lattice-based cryptography
- BKZ, lattice enumeration over Euclidean ring?
- Prove the output quality

Thanks for your attention!