The representation theory of the Jordanian algebra

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Abstract

We describe the complete set of pairwise non-isomorphic irreducible modules $S_\alpha$ over the algebra $R = k(x, y)/(xy - yx - y^2)$, and the rule how they could be glued to indecomposables. Namely, we show that $\text{Ext}^1_k(S_\alpha, S_\beta) = 0$, if $\alpha \neq \beta$. Also the set of all representations is described subject to the Jordan normal form of $Y$.

We study the properties of the image algebras in the endomorphism ring. Among facts we prove is that they are all basic algebras. Along this line we establish an analogue of the Gerstenhaber–Taussky–Motzkin theorem on the dimension of algebras generated by two commuting matrices. All image algebras of indecomposable modules turned out to be local complete algebras. We compare them with the Ringel’s classification by means of finding relations of image algebras. As a result we derive that all image algebras of $n$-dimensional representations with full block $Y$ are tame for $n \leq 4$ and wild for $m \geq 5$.

We suggest a stratification of representation space of $R$ by partitions of $n$ related to the Jordan normal form of $Y$. We give a complete classification by parameters for some strata and present examples of tame and even finite type (up to automorphisms) strata, while the generic stratum is wild.

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1 Introduction

We consider here a quadratic algebra given by the following presentation:
$R = k\langle x, y \rangle / (xy - yx - y^2)$. This algebra appeared in various different contexts in mathematics and physics. First of all it is a kind of a quantum plane: one of the two Auslander regular algebras of global dimension two in the Artin–Shelter classification [4]. The other one is a usual quantum plane $k\langle x, y \rangle / (xy - qyx)$. There were studied, for example, deformations of $GL(2)$ analogues to $GL_q(2)$ with respect to this algebra in 80-90th in Manin’s ’Quantum group’ [18], [16], where this algebra appeared under the name Jordanian algebra.

This algebra is also a simplest element in the class of RIT (relativistic internal time) algebras. The latter appeared and investigations were started in papers [3], [2], [1], [13], [5]. The class of RIT algebras arises from a modification of the Poincare algebra of the Lorenz group SO(3,1) by means of introducing the additional generator corresponding to the relativistic internal time. The algebra $R$ above is a RIT algebra of type (1,1). Our studies of this algebra reported in [12], [14].

Let we mention that algebra $R$ is a subalgebra of the first Weyl algebra $A_1$. The latter has no finite dimensional representations, but $R$ turned out to have quite a rich structure of them. Category of finite dimensional modules over $R$ contains, for example, as a full subcategory mod$GP(n,2)$, where $GP(n,2)$ is a Gelfand–Ponomarev algebra [7] with the nilpotency degrees of variables $x$ and $y$, $n$ and 2 respectively. On the other hand we show in section 5 that $R$ is residually finite dimensional.

We are interested here in representations over algebraically closed field $k$ of characteristic 0. In few places we suppose $k = \mathbb{C}$, this will be pointed
out separately. We denote throughout be the category of all $R$-modules by Mod$R$, the category of finite dimensional $R$-modules by mod$R$ and $\rho_n \in$ mod$R$ stands for an $n$-dimensional representation of $R$.

We study here the category mod$R$ of finite dimensional representations of infinite dimensional algebra $R$ first by encountering some properties of its finite dimensional images in the endomorphism ring End$(k^n)$. Then we suggest a stratification of a representation space related to partitions which define the Jordan structure of $Y = \rho_n(y)$ and give a classification and tameness results for some strata.

Toward the first approach we prove (section 2) that images in endomorphism rings are basic algebras, that is their semisimple parts are a direct sums of fields. This allow to associate a quiver to any representation and to classify representations using these quivers. It turns out that indecomposable modules have usually a typical wild quiver with one vertex and two loops and in some cases the quiver with one vertex and one loop. The simple, but important fact for the structural properties of image algebras is that $Y = \rho_n(y)$ is nilpotent for any $\rho_n \in$ mod$R$. Note that this is not necessarily the case when the characteristic of the basic field is not zero. After the description of all finite dimensional modules subject to Jordan normal form of $Y$ in section 4, we study irreducible and indecomposable modules in sections 6,7. We describe the complete set of irreducible modules $S_\alpha$ and show how one could glue them together: $\text{Ext}^1_k(S_\alpha, S_\beta) = 0$, if $\alpha \neq \beta$. This means that indecomposables have always $X$ with only one eigenvalue.

From the above results we see that any algebra which is an image of indecomposable representation $\rho_n$ is a local complete algebra. Hence we could apply Ringel’s classification [22] of local complete algebras to those images and after calculating defining relations of image algebras get that all of them are tame for $n \leq 4$ and for $n \geq 5$ they are wild. These results are described in section 10.

In section 8 we prove an analogue of the Gerstenhaber–Taussky–Motzkin theorem [8], [10] on the dimension of algebras generated by two commuting matrices. The dimension of image algebras of representations of $R$ does not exceed $n(n + 2)/4$ for even $n$ and $(n + 1)^2/4$ for odd $n$. This estimate is attained for the family of representations with the full block Jordan form of $Y$.

In section 9 we consider an action of $GL_n$ on the representation space of the algebra $R$: mod$(R,n)$, which is a representation space of a wild quiver with relation. We take as a model strata the one related to the full Jordan block $Y$ and show that it is parametrizable by two parameters in a conventional sense, and has a finite representation type with respect to auto-equivalence relation on reps (defined via gluing of isoclasses, which are coincide up to automorphisms of initial algebra, see section 5 for a precise definition). To get results on the auto-equivalence we describe the automorphism group of $R$ in section 3.2. We give examples of tame strata (in proper sense, e.i. not of finite representation type) up to auto-equivalence. Let we
mention, that normally, for generic partition, the stratum is wild for this algebra.

Main tool we use for the parametrization results is to consider in stead of the whole action of $GL_n$ on the representation space the action of the centralizer of Jordan form of $Y$ on those points of the space where $Y$ is in this Jordan form. While the group which acts is not reductive any more, the space where it acts become much simpler. Due to 1-1 correspondence of orbits under these two actions one can lift results on classification from one setting to another.

2 Structural properties of the images of representations and quivers

We consider here the case when $k$ is an algebraically closed field of characteristic zero. Let $\rho : R \to \text{End}(k^n)$ be an arbitrary finite dimensional representation of $R$, denote by $A_{\rho,n} = \rho(R)$ an image of $R$ in the endomorphism ring. We will write also $A_n$ or $A$ when it is clear from the context which $\rho$ and $n$ we mean.

We derive in this section some structural properties of algebras $A_{n,\rho}$. They all turned out to be basic; in any of such algebra the image of $y$ is nilpotent; complete system of orthogonal idempotents in $A$ corresponds to the different eigenvalues of $x$, etc.

Let $J(A) = J$ be the Jacobson radical of the algebra $A_{\rho,n}$. We show first that $A$ is basic, that is its semisimple part $A/J(A)$ is a direct product of division rings, or in our case of algebraically closed field $k$ it is the same as direct product of several copies of the field $k$. Basic algebras take their name particularly because they are basic from the point of view of Morita equivalence. Due to the Wedderburn–Artin theorem and the equivalence of categories of modules $\text{Mod}-R$ and $\text{Mod}-R_n$ ($R_n$ are $n \times n$ matrices over $R$), any artinian semisimple algebra is Morita equivalent to a finite direct sum of division rings. So we show that image algebras of all finite dimensional representations of $R$ has this special place between all finite dimensional algebras in sense of Morita equivalence. We also prove in section 5 that $R$ is residually finite dimensional, and together with the above fact it gives as a consequence that $R$ is residually basic.

Lemmata below describes the structural properties of image algebras for $R$. The following fact probably allow many different proofs. We present here the shortest we know.

**Lemma 2.1** Let $Y = \rho_n(y)$. Then the matrix $Y$ is nilpotent.

**Proof.** Suppose that the matrix $Y$ is not nilpotent and hence has a nonzero eigenvalue. We take a projector $P$ on the subspace corresponding to this eigenvalue. It is obviously commute with any matrix, particularly with $Y : PY = YP$, and is an idempotent operator: $P^2 = P$. Hence multiplying
our relation $XY - YX = Y^2$ from the right and from the left hand side by $P$ and using above two notices we can observe that operators $X' = PXP$ and $Y' = PYP$ also satisfy the same relation: $X'Y' - Y'X' = Y'^2$. Taking into account that $Y'$ has a form of one or more Jordan blocks with the same nonzero eigenvalue $\lambda$, we get that traces of right and left parts of the relation can not coincide. This contradiction complete the proof. □

Let we prove here also a little bit more general fact.

**Lemma 2.2** Let $X, Y$ be $n \times n$ matrices over an algebraically closed field $k$ of characteristic zero. Assume that the commutator $Z = XY - YX$ commutes with $Y$. Then $Z$ is nilpotent.

**Proof.** Assume the contrary. Then $Z$ has a non-zero eigenvalue in $z \in k$. Let $L = \ker (Z - zI)^n$ and $N = \text{Im}(Z - zI)^n$.

The subspace $L$ is known as a main subspace for $Z$ corresponding to the eigenvalue $z$. Clearly $L \neq \{0\}$. It is well-known that $k^n$ is the direct sum $k^n = L \oplus N$ of $Z$-invariant linear subspaces $L$ and $N$. Due to $ZY = YZ$, the subspaces $L$ and $N$ are also invariant for $Y$.

Consider the linear projection $P$ along $N$ onto $L$. Since $L$ and $N$ are invariant under both $Y$ and $Z$, we have $ZP = PZ$ and $YP = PY$. Multiplying the equality $Z = XY - YX$ by $P$ from the left and from the right hand side and using the equalities $ZP = PZ$, $YP = PY$ and $P^2 = P$, we get

$$ZP = PXPY - YPXP.$$ 

Since $ZP$ vanishes on $N$ and $ZP - zI$ has only one eigenvalue $z$, then after restriction to $L$, we have $\text{tr } ZP = z \dim L$. On the other hand $\text{tr } PXPY = \text{tr } YPXP$ since the trace of a product of two matrices does not depend on the order of the product. Thus, the last display implies that $z \dim L = 0$, which is not possible since $z \neq 0$ and $\dim L > 0$. □

Coming back to the case of simplest RIT algebra, we have further

**Lemma 2.3** Let $X = \rho_n(x)$. Then the matrix $S = S(X) = (X - \lambda_1 I) \ldots (X - \lambda_r I)$ is nilpotent.

**Proof.** Note that $\text{Spec } p(X) = p(\text{Spec } X)$ for any polynomial $p$. $\text{Spec } X$ in our case is $\{\lambda_1, \ldots, \lambda_r\}$ and hence $\text{Spec } S = \{0\}$. Therefore the matrix $S$ is nilpotent. □

**Lemma 2.4** Any nilpotent element of the algebra $A = \rho(R)$ belongs to the radical $J(A)$.

**Proof.** We will use here the feature of an algebra $A$ that it has the presentation as a quotient of free algebra containing our main relation. Namely, it has a presentation: $A = k\langle x, y | xy - yx = y^2, R_A \rangle$, where $R_A \subset k\langle x, y \rangle$ is the set of additional relations specific for the given image algebra. Thus
we can think of elements in \( A \) as of polynomials in two variables (subject to some relations). Let \( Q(x) \) be a polynomial on one variable \( Q(x) \in k[x] \) and \( Q(X) \in A \) be a nilpotent element with the degree of nilpotency \( N: Q^N = 0 \). We show first that \( Q \in J(A) \). We have to check that for any polynomial \( a \in k(x, y) \), \( 1 - a(X, Y)Q(X) \) is invertible. It suffices to verify that \( a(X, Y)Q(X) \) is nilpotent. By lemma 2.1 \( Y \) is nilpotent. Denote by \( m \) the degree of nilpotency of \( Y: Y^m = 0 \). Let we verify that \( (a(X, Y)Q(X))^{mN} = 0 \). Present \( a(X, Y) \) as \( u(X) + Yb(X, Y) \). If then we consider a word of length not less then \( mN \) on letters \( \alpha = u(X)Q(X) \) and \( \beta = Yb(X, Y)Q(X) \) then we can see that it is equal to zero. Indeed, if there are at least \( m \) letters \( \beta \) then using the relation \( XY - YX = Y^2 \) to commute the variables one can rewrite the word as a sum of words having a subword \( Y^m \). Otherwise our word has the subword \( \alpha^N = u(X)^N Q(X)^N = 0 \). Thus, \( Q(X) \in J(A) \).

Note now that if we have an arbitrary nilpotent polynomial \( G(X, Y) \), we can separate the terms containing \( Y: G(X, Y) = Q(X) + YH(X, Y) \). To obtain nilpotency of any element \( a(X, Y)G(X, Y) \) it suffices to verify nilpotency of \( a(X, Y)Q(X) \), which was already proven, because the relation \( [X, Y] = Y^2 \) allows to commute with \( Y \), preserving (or increasing) the degree of \( Y \).

**Corollary 2.5** The Jacobson radical of \( A = \rho(R) \) consists precisely of all nilpotent elements.

Particularly,

**Corollary 2.6** Let \( Y = \rho(y) \). Then \( Y \in J(A) \).

Let we formulate here also another property of the radical, which will be on use later on.

**Corollary 2.7** The Jacobson radical of \( A = \rho(R) \) consists of all polynomials on \( X = \rho(x) \) and \( Y = \rho(y) \) without constant term if and only if \( X \) is nilpotent in \( A \).

**Proof.** In one direction this is trivial, we should ensure only that if \( X^N = 0 \) then \( p(X, Y)^{2N} = 0 \) for any polynomial \( p \) such that \( p(0, 0) = 0 \) using the relation \( XY - YX = Y^2 \) which is an easy check. \( \square \)

**Theorem 2.8** Let \( A_{\rho,n} \) be the image algebra of \( R = k(x, y)/(xy - yx - y^2) \) under the \( n \)-dimensional representation \( \rho \) and \( X = \rho_n(x), \ Y = \rho_n(y) \) be its generators. Then \( A/J \) is a commutative one-generated ring \( k[x]/S(x) \), where \( S(x) = (x - \lambda_1)\ldots(x - \lambda_k) \) and \( \lambda_1, \ldots, \lambda_k \) are all different eigenvalues of the matrix \( X \).

**Proof.** From the corollary 2.6 we can see that \( A/J \) is an algebra of one generator \( x \): \( A/J \simeq k[x]/I \). We are going to find now an element which generates the ideal \( I \).
First of all by lemmas 2.3 and 2.4 \( S \in J(A) \), hence \( S(x) = (x - \lambda_1) \ldots (x - \lambda_r) \in I \). Let we show now that \( S \) divides any element of \( I \). If some polynomial \( p \in k[x] \) does not vanish in some eigenvalue \( \lambda \) of \( X \) then \( p(X) \notin J(A) \). Indeed, the matrix \( p(X) \) has a non-zero eigenvalue, than \( p(\lambda) \neq 0 \) and hence \( I - \frac{1}{p(\lambda)}p(X) \) is non-invertible. Therefore \( p(X) \notin J(A) \). Thus, \( S(x) \) is the generator of \( I \). This finishes the proof. \( \Box \)

**Corollary 2.9** The system \( e_i = p_i(X)/p_i(\lambda_i) \), where

\[
p_i(X) = (X - \lambda_1 I) \ldots (X - \lambda_i I) \ldots (X - \lambda_r I)
\]

and \( \lambda_i \) are a different eigenvalues of \( X = \rho(x) \) is a complete system of orthogonal idempotents of \( A/J \).

**Proof.** Orthogonality of \( e_i \) is clear from the presentation of \( A/J \) as \( k[x]/\text{id}(S) \) proven in theorem 2.8. \( \Box \)

**Theorem 2.10** For any finite dimensional representation \( \rho \) the semisimple part of \( A_\rho \) is a product of a finite number of copies of the field \( k \):

\[
A/J = \prod_{i=1}^{r} k_i,
\]

where \( r \) is the number of different eigenvalues of the matrix \( X = \rho(x) \).

**Proof.** We shall construct an isomorphism of \( A/J \) and \( \prod_{i=1}^{r} k_i \) using the system \( e_i, i = 1, \ldots, r \) of idempotents from the corollary 2.9. Clearly \( e_i \) form a basis of \( A/J \) as a linear space over \( k \). From the presentation of \( A/J \) as a quotient \( k[x]/\text{id}(S) \) given in the theorem 2.8 it is clear that the dimension of \( A/J \) is equal to the degree of polynomial \( S(x) \), which coincides with the number of different eigenvalues of the matrix \( X \). Since idempotents \( e_i \) are orthogonal, they are linearly independent and therefore form a basis of \( A/J \). The multiplication of two arbitrary elements \( a, b \in A/J, a = a_1 e_1 + \ldots + a_r e_r, b = b_1 e_1 + \ldots + b_r e_r \) is given by the formula \( ab = a_1 b_1 e_1 + \ldots + a_r b_r e_r \) due to orthogonality of the idempotents \( e_i \). Hence the map \( a \mapsto (a_1, \ldots, a_r) \) is the desired isomorphism of \( A/J \) and \( \prod_{i=1}^{r} k_i \). \( \Box \)

Since all images turned out to be basic algebras we can associate to them a *quiver* in a conventional way (see, for example, \([9], [11]\)). The vertices will correspond to the idempotents \( e_i \) or by the corollary 2.9 equivalently, to the different eigenvalues of matrix \( X \). The number of arrows from vertex \( e_i \) to the vertex \( e_j \) is the \( \dim_k e_i (J/J^2) e_j \). There are a finite number of such quivers in fixed dimension \( n \) (the number of vertices bounded by \( n \), the number of arrows between any two vertices roughly by \( n^2 \)).

Let we prove now the following lemma. Denote by \( \hat{Y} \) the image of \( Y \) under the factorization by the square of radical: \( \hat{Y} = \varphi Y \), for \( \varphi : A \to A/J^2 \).
Lemma 2.11 If in the representation $\rho : R \to A$, $X = \rho(x)$ has only one eigenvalue $\lambda$, then the corresponding quiver $Q_A$ has one vertex and number of loops is a dimension of the vector space $\text{Sp}_k\{\bar{X} - \lambda I, \bar{Y}\}$, which does not exceed 2.

Proof. Due to the description of idempotents above in the case of one eigenvalue the only idempotent is unit. Hence we have to calculate $\dim_k J/J^2$, where $J = \text{Jac}(A)$. Since $X - \lambda I$ satisfy the same relation as $X$ we could apply the corollary 2.7 and result immediately follows. \qed

After we have proved that all image algebras are basic we can define an equivalence relation on representations of RIT algebra using quivers of its images.

Definition 2.12 Two representations $\rho_1$ and $\rho_2$ of the algebra $R$ are quiver-equivalent $\rho_1 \sim Q \rho_2$ if quivers associated to algebras $\rho_1(R)$ and $\rho_2(R)$ coincide.

As an example let us clarify the question on how many quiver-equivalence classes appear in the family of representations

$$\mathcal{M}_n = \{(X,Y) \in \text{mod}(R,n) | \text{rk} Y = n - 1\}$$

and which quivers are realized.

Proposition 2.13 For any $n \geq 3$ families of representations $\mathcal{M}_n$ belong to one quiver-equivalence class. Corresponding quiver consists of one vertex and two loops.

Proof. This will directly follow from Lemma 2.11, when we ensure in section 4 that $X$ has only one eigenvalue in the family $\mathcal{M}_n$ and take into account that when we have full block $Y$, the dimension of the linear space $\text{Sp}_k\{\bar{X} - \lambda I, \bar{Y}\}$ can not be smaller then 2. \qed

3 Automorphisms of RIT algebras and multiplication formulas

Here we intend to describe the group of automorphisms of the RIT algebra $R$ in order to use this information later on for the classification results. It turned out to be quite small, compared with automorphisms of the first Weyl algebra $A_1$, which contains $R$ as a subalgebra. Automorphisms of the $A_1$ were described in [17], the case of an arbitrary Weyl algebra $A_n$ was discussed in [15].

First we shall prove lemmata on multiplication in RIT, it will be on use for various purposes later on.
3.1 Preliminary facts on multiplication in RIT

Since the defining relation for $R$: $xy = yx + y^2$ form a Gröbner basis with respect to the ordering $x > y$, the basis of our algebra as a vector space over $k$ consists of the monomials $y^kx^l$, $k, l = 0, 1, \ldots$. These are those monomials which do not contain the highest term $xy$ of the defining relation.

We prefer to show this here in a canonical way. For this we shall remain the definition of a Gröbner basis of an ideal and the method of construction of a linear basis of an algebra given by relations, based on the Gröbner basis technique. Using this canonical method it could be easily shown that, for example, some Sklyanin algebras enjoys a PBW property. This was proved in [20], the arguments there are very intelligent and interesting in their own right, but quite involved.

Let $A = k\langle X \rangle/I$. The first essential step is to fix an ordering on the semigroup $\beta = \langle X \rangle$. We fix some linear ordering in the set $X$. Then we have to extend it to an admissible ordering on $\beta$, i.e. it has to satisfy the conditions:

1) if $u, v, w \in \beta$ and $u < v$ then $uw < vw$ and $wu < vw$

2) the descending chain condition (d.c.c.): there is no infinite properly descending chain of elements of $\beta$.

We shall use the degree-lexicographical ordering in the semigroup $\beta$, namely for arbitrary $u = x_{i_1} \ldots x_{i_n}, v = x_{j_1} \ldots x_{j_k} \in \beta$ we say $u > v$, when either $\deg u > \deg v$ or $\deg u = \deg v$ and for some $l$: $x_{i_l} > x_{j_l}$ and $x_{i_m} = x_{j_m}$ for any $m < l$. This ordering is admissible.

Denote by $\bar{f}$ the highest term of polynomial $f \in A = k\langle X \rangle$ with respect to the above order.

**Definition 4.2.** Subset $G \in I, I \triangleleft \langle X \rangle$ is a Gröbner basis of an ideal if the set of highest terms of elements of $G$ generates the ideal of highest terms of $I$: $id\{\bar{G}\} = \bar{I}$.

**Definition 4.3.** We will say that monomial $u \in \langle X \rangle$ is normal if it does not contain as a submonomial any highest term of an element of the ideal $I$.

From these two definitions it is clear that normal monomial is a monomial which does not contain any highest term of an element of Gröbner basis of the ideal $I$. If Gröbner basis turned out to be finite then the set of normal words is constructible.

In case when an ideal $I$ of defining relations for $A$ has a finite Gröbner basis, the algebra called standardly finitely presented (s.f.p.).

It is easy, but useful fact that $\langle X \rangle$ is isomorphic to the direct sum $I \oplus \langle N \rangle_k$ as a linear space over $k$, where $\langle N \rangle_k$ is the linear span of the set of normal monomials from $\langle X \rangle$ with respect to the ideal $I$. We claim also (without check, which is not difficult) that the set of normal words form a linear basis. Hence given a Gröbner basis $G$ of an ideal $I$, we can construct a linear basis of an algebra $A = \langle X \rangle/I$ as a set of normal (with respect to $I$) monomials, at least in case when $A$ is s.f.p.

As a consequence we immediately get the following
Lemma 3.1 The system of monomials $y^n x^m$ form a basis of algebra $R$ as a vector space over $k$.

We say that an element is in normal form, if it is presented as a linear combination of normal monomials. After we have a linear basis of normal monomials we should know how to multiply them to get again an element in normal form.

Now we are going to prove the following lemmata, where we express precisely normal forms of some products.

Lemma 3.2 The normal form of the monomial $xy^n$ in algebra $R$ is the following:
\[ xy^n = y^n x + ny^{n+1}. \]

Proof. This can be proven by induction on $n$. The case $n = 1$ is just our algebra’s relation. Suppose $n > 1$ and the equality $xy^{n-1} = y^{n-1}x + (n-1)y^n$ holds. Multiplying it by $y$ from the right and reducing by the relation $xy -yx = y^2$, we obtain
\[ xy^n = y^{n-1}xy + (n-1)y^{n+1} = y^n x + y^{n+1} + (n-1)y^{n+1} = y^n x + ny^{n+1}. \]

The proof is now complete. □

Lemma 3.3 The normal form of the monomial $x^ny$ in algebra $R$ is the following: $x^ny = \sum_{k=1}^{n+1} \alpha_{k,n}y^k x^{n-k+1}$, where $\alpha_{k,n} = n!/(n-k+1)!$ for $k = 1, \ldots, n+1$.

Proof. We are going to prove this formula inductively using the previous lemma. As a matter of fact we shall obtain recurrent formulas for $\alpha_{k,n}$. In the case $n = 1$ the relation $xy -yx = y^2$ implies the desired formula with $\alpha_{1,1} = \alpha_{2,1} = 1$. Suppose $n$ is a positive integer and there exist positive integers $\alpha_{k,n}, k = 1, \ldots, n+1$ such that $x^ny = \sum_{k=1}^{n+1} \alpha_{k,n}y^k x^{n-k+1}$. Multiplying the latter equality by $x$ from the left and using lemma 3.2 we obtain
\[ x^{n+1}y = \sum_{k=1}^{n+1} \alpha_{k,n}xy^k x^{n-k+1} = \sum_{k=1}^{n+1} \alpha_{k,n}y^k x^{n-k+2} + \sum_{k=1}^{n+1} \alpha_{k,n}ky^{k+1} x^{n-k+1}. \]

Rewriting the second term as $\sum_{k=1}^{n+2} \alpha_{k-1,n}(k-1)y^k x^{n-k+2}$ (here we assume that $\alpha_{0,n} = 0$), we arrive to
\[ x^{n+1}y = \sum_{k=1}^{n+2} \alpha_{k,n+1}y^k x^{n-k+2}, \]

where $\alpha_{k,n+1} = \alpha_{k,n} + (k-1)\alpha_{k-1,n}$ for $k = 1, \ldots, n+1$ and $\alpha_{n+2,n+1} = (n+1)\alpha_{n+1,n}$. 10
Let us prove now the formula for $\alpha_{k,n}$. For $n = 1$ it is true since $\alpha_{1,1} = \alpha_{1,2} = 1$. Then we use inductive argument. Suppose the formula is true for $n$. We are going to apply the recurrent formula appeared above:

$$
\alpha_{k,n+1} = \alpha_{k,n} + (k-1)\alpha_{k-1,n} = \frac{n!}{(n-k+1)!} + (k-1)\frac{n!}{(n-k+2)!} = \frac{(n+1)!}{(n-k+2)!}
$$

and the formula is verified for $1 \leq k \leq n+1$. For $k = n+2$, we have $\alpha_{n+2,n+1} = (n+1)\alpha_{n+1,n} = (n+1)! = (n+1)!$. This completes the proof.

\[\square\]

### 3.2 Automorphisms of RIT algebras

We are going to describe the automorphism group of the simplest RIT algebra here. We shall prove.

**Theorem 3.4** All automorphisms of $R = k\langle x, y \mid xy - yx = y^2 \rangle$ are of the form $x \mapsto \alpha x + p(y)$, $y \mapsto \alpha y$, where $\alpha \in k \setminus \{0\}$ and $p \in k[y]$ is a polynomial on $y$. Hence the group of automorphisms isomorphic to a semidirect product of an additive group of polynomials $k[y]$ and a multiplicative group of the field $k^*$: $\text{Aut}(R) \cong k[y] \rtimes k^*$.

**Proof.** Key observation for this proof is that in our algebra there exists a minimal ideal with commutative quotient. Namely, the two-sided ideal $J$ generated by $y^2$.

**Lemma 3.5** If the quotient $R/I$ is commutative then $y^2 \in I$ (that is $J \subseteq I$).

**Proof.** The images of $x$ and $y$ in this quotient commute. Hence

$$
0 = (x + I)(y + I) - (y + I)(x + I) = xy - yx + I = y^2 + I.
$$

Therefore $y^2 \in I$. \[\square\]

The property of an ideal to be a minimal ideal with commutative quotient is invariant under automorphisms.

Let us denote by $\tilde{y} = f(x, y)$ the image of $y$ under an automorphism $\varphi$. Then the ideal generated by $\tilde{y}^2$ coincides with the ideal generated by $y^2$: $J = (y^2) = (\tilde{y}^2)$.

Using the property of multiplication in $R$ from lemma 3.3, we can see that two-sided ideal generated by $y^2$ coincides with the left ideal generated by $y^2$: $Ry^2R = y^2R$. Indeed, let us present an arbitrary element of $Ry^2R$ in the form $\sum a_i y^2 b_i$, where $a_i$, $b_i \in R$ are written in the normal form $a_i = \sum \alpha_{k,i} y^k x^l$, $b_i = \sum \beta_{k,i} y^k x^l$. Using the relations from lemma 3.2, we can pull $y^2$ to the left through $a_i$’s and get the sum of monomials, which all contain $y^2$ at the left hand side. Thus, $\sum a_i y^2 b_i = y^2 u$, $u \in R$.

Obviously automorphism maps the one-sided ideal $y^2R$ onto the one-sided ideal $\tilde{y}^2R$, both of which coincide with $J = (y^2) = (\tilde{y}^2)$. From this we obtain a presentation of $y^2$ as $\tilde{y}^2 u$ for some $u \in R$. Considering usual
degrees of these polynomials (on the set of variables $x, y$), we get $2 = 2k + l$, where $k = \deg \tilde{y}$ and $l = \deg u$. Obviously $k \neq 0$. Hence the only possibility is $k = 1$ and $l = 0$.

Thus, $\varphi(y) = \tilde{y} = \alpha x + \beta y + \gamma$ and $u = c$ for some $\alpha, \beta, \gamma, c \in k$. Substituting these expressions into the equality $y^2 = \tilde{y}^2 u$, we get $c(\alpha x + \beta y + \gamma)^2 = y^2$. Comparing the coefficients of the normal forms of the right and left hand sides of this equality, we obtain $\alpha = \gamma = 0$, $\beta \neq 0$. Hence

$\varphi(y) = \beta y$.

Now we intend to use invertibility of $\varphi$. Due to it there exists $\alpha_{ij} \in k$ such that $x = \sum \alpha_{ij} \tilde{y}^i \tilde{x}^j$. Substituting $\tilde{y} = \beta y$, we get $x = \sum_{r=0}^{N} p_r(y) \tilde{x}^r$,

where $N$ is a positive integer, $p_r \in k[y]$ and $p_N \neq 0$. Comparing the degrees on $x$ of the left and right hand sides of the last equality we obtain $1 = kN$, where $k = \deg_x \tilde{x}$. Hence $k = N = 1$, that is $x = p_0(y) + p_1(y) \tilde{x}$ and $\tilde{x} = q_0(y) + q_1(y)x$, where $p_0, p_1, q_0, q_1 \in k[y]$. Substituting $\tilde{x} = q_0(y) + q_1(y)x$ into $x = p_0(y) + p_1(y) \tilde{x}$, we obtain $q_1 \in k$, that is $\tilde{x} = cx + p(y)$ for $c \in k$.

One can easily verify that the relation $\tilde{x} \tilde{y} - \tilde{y} \tilde{x} = \tilde{y}^2$ is satisfied for $\tilde{x} = cx + p(y)$, $\tilde{y} = \beta y$ if and only if $c = \beta$. This gives us the general form of the automorphisms: $\tilde{x} = cx + p(y)$, $\tilde{y} = cy$, $c \neq 0$.

Now we see that the group of automorphisms is a semidirect product of the normal subgroup isomorphic to the additive group of polynomials $k[y]$ and the subgroup isomorphic to the multiplicative group $k^*$. The precisely written formula for multiplication in $\text{Aut} R$ is the following:

$\varphi_1 \varphi_2 = (p_1(y), c_1)(p_2(y), c_2) = (c_2 p_1(y) + p_2(c_1 y), c_1 c_2)$

for $\varphi_1, \varphi_2 \in \text{Aut} R$. □

4 Irreducible modules, description of all finite dimensional modules

We intend to prove here the following

**Theorem 4.1** The description of the complete set of finite dimensional representations of $R$ (subject to the Jordan form of $Y$) are given by
where partition on blocks in $X_n$ correspond to the partition defined by the Jordan form of $Y_n$. Diagonal blocks of $X_n$ are matrices $X_0^n + T$, where $X^0$ is a matrix with the vector $[0, 1, 2, ...]$ on the first upper diagonal and zeros elsewhere. $T$ is an arbitrary upper diagonal Toeplitz matrix. All the rest of blocks of $X$ are upper diagonal rectangular Toeplitz matrices.

From this theorem immediately follows a precise description of all irreducible and completely reducible modules.

**Corollary 4.2** A complete set of pairwise non-isomorphic finite dimensional irreducible $R$-modules is \{ $S_a | a \in k$ \}, where $S_a$ defined by the following action of $X$ and $Y$ on one-dimensional vector space: $Xu = \alpha u, Y u = 0$.

All completely reducible representations are given by matrices: $Y_n = (0)$, $X_n$ is a diagonal matrix $\text{diag}(a_1, ..., a_n)$.

**Proof.** Let we describe an arbitrary representation $\rho_n : R \to M_n(k)$ of $R$, for $n \in \mathbb{N}$. We can assume that the image of one of the generators $Y = \rho_n(y)$ is in normal Jordan form.

**Full Jordan block case.**

Let us first find all possible matrices $X = \rho_n(x)$ in the case when $Y$ is just full Jordan block: $Y = J_n$. We have to find than matrices $X = (a_{ij})$ satisfying the relation $[X, J_n] = J_n^2$. Let $B = [X, J_n] = (b_{ij})$, then $b_{ij} = a_{i+1,j} - a_{i,j-1}$. From the condition $B = J_n^2$ it follows that $b_{ij} = 0$ if $i \neq j - 2$ and $b_{ij} = 1$ if $i = j - 2$. Here and later on we will use the following numeration of diagonals: main diagonal has number 0, upper diagonals have positive numbers $1, 2, ..., n - 1$ and lower diagonals have negative numbers $-1, -2, ..., -n + 1$: 

$$
\begin{pmatrix}
\bullet & n-1 \\
\vdots \\
1 & \\
-2 & \bullet & n+1 \\
\end{pmatrix}
$$
The first condition above means than that in the matrix $X$ elements of any diagonal with number $0 \leq k \neq 1$ coincide and are zero for $k < 0$. From the second condition it follows that the elements of the first upper diagonal form an arithmetic progression with difference 1: $a + 1, \ldots, a + n - 1$.

Therefore we have the following sequence of representations:

$$Y_n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad X_n = \begin{pmatrix} \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots \end{pmatrix} \quad (2)$$

Here and below we will draw a diagonal as a continuous line if all its elements coincide and as a thick line if its elements form an arithmetic progression with difference one.

Denote by $X^0$ a matrix with the sequence $0, 1, 2, \ldots$ on the first diagonal and zeros elsewhere. Then our family of representations consists of pairs of matrices $(X_n, Y_n) = (X^0 + T, J_n)$, where $T$ is an arbitrary upper diagonal Toeplitz matrix. Let we remind that upper diagonal (rectangular) Toeplitz matrix is a matrix with entries $a_{ij}$ defined only by the difference $i - j$. It has zeros below the main diagonal (or upper main diagonal in a proper rectangular case).

Note that one could get a clue on what the set of representations is from the following observation. First, the matrix $X^0$ satisfies the relation $[X^0, Y] = Y^2$ for $Y = J_n$. On the other hand a matrix $X = X^0 + M$ satisfies the relation $[X, Y] = Y^2$ if and only if $M$ commutes with $Y = J_n$. Any matrix having only one non-zero diagonal with equal elements on it commutes with $Y = J_n$. Hence we have at least all linear combinations of those matrices in the set of representations, additional arguments as above show that there are no others.

*The case of an arbitrary partition.*

Consider now the general case when the Jordan normal form of $Y$ contains several Jordan blocks: $Y = (J_1, \ldots, J_m)$.

Cut an arbitrary matrix $X$ into the square and rectangular blocks of corresponding size, denote the blocks by $A_{ij}, i, j = 1, m$.

Then we can describe the structure of the matrix $B = [X, Y]$ in the following way:
\[ B = \begin{bmatrix} [A_{11}] & [A_{12}] & [A_{1m}] \\ & [A_{22}] & \vdots \\ & & \ddots \\ [A_{m1}] & [A_{m2}] & [A_{mm}] \end{bmatrix}, \text{ where } [A_{ij}] = A_{ij}J_i - J_jA_{ij}. \]

From the condition \( B = Y^2 \) we have that \([A_{ii}, J_i] = J_i^2 \) and hence \( A_{ii} \) is the same as in the previous case when \( Y \) was just a full Jordan block and \( A_{ij}J_i - J_jA_{ij} = 0 \) for \( i \neq j \). The latter condition means that \( A_{ij} \) for \( i \neq j \) has the following structure.

\[
\begin{bmatrix} \vdots \\ \vdots \end{bmatrix}
\text{ or }
\begin{bmatrix} \vdots \\ \vdots \end{bmatrix}
\]

The elements of any diagonal here marked as a line are equal to each other, elements of different diagonals could be different and they are equal to zero below the upper diagonal of maximal length (the matrix is non-square in general). As a result we have the family of representations described in the theorem 4.1. □

5 \( R \) is residually finite dimensional

Let us consider now one of the sequences of representations constructed in the previous section: \( \varepsilon_n : R \to \text{End } k^n \), defined by \( \varepsilon_n(y) = J_n, \varepsilon_n(x) = X_n^0 \) as above. Note that this sequence is basic in the following sense. As was actually shown in 4, all representations (2) corresponding to \( Y \) with one Jordan block could be obtained from \( \varepsilon_n \) by the following automorphism of \( R \), \( \varphi : R \to R : x \mapsto x + a, y \mapsto y \) where \( a \in R \) such that \([a, y] = 0\).

In addition to the conventional equivalence relation on the representations given by simultaneous conjugation of matrices: \( \rho' \sim \rho'' \) if there exists \( g \in GL(n) \) such that \( g\rho' g^{-1} = \rho'' \) or equivalently, \( R \)-modules corresponding to \( \rho' \) and \( \rho'' \) are isomorphic, we introduce here one more equivalence relation.
**Definition 5.1** We say that two representations of the algebra $R$ are auto-equivalent (equivalent up to automorphism) $\rho' \sim_A \rho''$ if there exists $\varphi \in \text{Aut}(R)$ such that $\rho' \varphi \sim \rho''$.

So we can state that any full block representation is auto-equivalent to $\varepsilon_n$ for appropriate $n$.

We will prove now that the sequence of representations $\varepsilon_n$ asymptotically is faithful.

Start with the calculation of matrices which are image of monomials $y^kx^m$ under representation $\varepsilon_n$.

**Lemma 5.2** For the representation $\varepsilon$ as above the matrix $\varepsilon(y^kx^m)$ has the following shape: there is only one nonzero diagonal, number $k+m$, in the above numeration, where appears the sequence $p(0), p(1), \ldots, p(j), \ldots$ of values of degree $m$ polynomial $p(j) = (k+j) \ldots (k+m+j-1) = \prod_{i=1}^{m}(k+j+i)$.

**Proof.** Image $\varepsilon(x^m)$ of the monomial $x^m$ is a matrix with vector $[1 \cdot 2 \cdot \ldots \cdot m, 2 \cdot 3 \cdot \ldots \cdot (m+1), \ldots]$ on the (upper) diagonal number $m$ in the above numeration and zeros elsewhere. Multiplication by $\varepsilon(y^k)$ acts on matrix by moving up all rows on $k$ steps. We can now see that matrix corresponding to the polynomial $y^kx^m$ can have only one nonzero diagonal, number $m+k$, and vector in this diagonal is the following: $[(k+1) \ldots (m+k), (k+2) \ldots (m+k+1), \ldots]$.

**Theorem 5.3** Let $\varepsilon_n$ be the sequence of representations of $R$ as above. Then $\bigcap_{n=0}^{\infty} \ker \varepsilon_n = 0$.

**Proof.** We are going to show that $\varepsilon_n(f) \neq 0$ for $n \geq 2 \deg f$. Suppose that $n$ is sufficiently large and $\varepsilon_n(f)$ is zero and get a contradiction. Denote by $l$ degree of polynomial $f$, and let $f = f_1 + \ldots + f_l$ be a decomposition of $f \in R$ on the homogeneous components of degrees $i = 1, \ldots, l$ respectively. From lemma 5.2 we know now how the matrix which is an image of an arbitrary monomial $y^kx^m$ looks like.

Applying the lemma 5.2 to each homogeneous part of the given polynomial $f$ we get

$$f_i = \sum_{k+m=l} a_{k,m}y^kx^m = \sum_{r=0}^{l} a_r y^{l-r}x^r$$

is a sum of matrices $\sum_{r=0}^{l} a_r M_r$, where $M_r$ has the vector $[(p(0), \ldots, p(j))$:

$$\left(\prod_{i=1}^{r} (l-r+i), \prod_{i=1}^{r} (l-r+i+1), \ldots, \prod_{i=1}^{r} (l-r+i+j), \ldots\right)$$

on the diagonal number $l$ (all other entries are zero). The number on the $j$-th place of this diagonal is the value in $j$ of the polynomial

$$P(j) = (l-r+j) \cdot \ldots \cdot (l+j-1)$$
of degree exactly $r$. Therefore the sum $\sum_{r=0}^{l} a_r M_r$ has a polynomial on $j$ of degree $N = \max\{r : a_r \neq 0\}$ on the diagonal number $l$. Since any polynomial of degree $N$ has at most $N$ zeros we arrive to a contradiction in the case when $l$th diagonal has length more than $l$. Hence for any $n \geq 2 \deg f$, $\varepsilon_n(f) \neq 0$. \hfill $\square$

Let we recall that an algebra $R$ residually has some property $P$ means that there exists a system of equivalence relations $\tau_i$ on $R$ with trivial intersection, such that in the quotient of $R$ by any $\tau_i$ property $P$ holds.

From the Theorem 5.3 we have the following corollary considering equivalence relations modulo ideals $\ker \varepsilon_n$.

**Corollary 5.4** Algebra $R$ is residually finite dimensional.

## 6 Indecomposable modules

**Lemma 6.1** Let $M = (X, Y)$ be a (finite dimensional) indecomposable module over $R = k\langle x, y | xy - yx = y^2 \rangle$. Then $X$ has a unique eigenvalue.

**Proof.** Denote by $M^X_\lambda$ the main eigenspace for $X$ corresponding to its eigenvalue $\lambda$: $M^X_\lambda = \bigcup_{k=0}^{\infty} \ker (X - \lambda I)^k$. Obviously $M^X_\lambda = \ker (X - \lambda I)^m$, where $m$ is the maximal size of blocks in the Jordan normal form of $X$. It is well-known that $M = \bigoplus_i M^X_{\lambda_i}$, where the direct sum is taken over all different eigenvalues $\lambda_i$ of $X$. We shall show that $M^X_{\lambda_i}$ are in fact $R$-submodules.

Let $u \in M^X_\lambda$, that is $(X - \lambda I)^m u = 0$. We calculate $(X - \lambda I)^n Y u$ for arbitrary $n$. Using the fact that the mapping defined on generators $\varphi(x) = x - \lambda, \varphi(y) = y$ extends to an automorphism of $R$ (see 3.2), we can apply it to the multiplication formula from Lemma 3.3 to get $(x - \lambda)^n y = \sum_{k=1}^{n+1} y^k (x - \lambda)^{n-k+1}$. Taking into account that $Y^l = 0$ for some positive integer $l$, we can choose $N$ big enough, for example $N \geq m + l$, such that

$$(X - \lambda I)^N Y u = \sum_{k=1}^{N+1} \alpha_{k,N} Y^k (X - \lambda I)^{N-k+1} u = 0$$

either due to $(X - \lambda I)^{N-k+1} u = 0$ or due to $Y^k = 0$.

This shows that $Y u \in M^X_\lambda$, that is $M^X_\lambda$ is invariant with respect to $Y$. \hfill $\square$

As an immediate corollary we have the following.

**Proposition 6.2** Any finite dimensional $R$-module $M$ decomposes into the direct sum of submodules $M^X_{\lambda_i}$ corresponding to different eigenvalues $\lambda_i$ of $X$. 

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Corollary 6.3 Let $M$ be indecomposable module corresponding to the representation $\rho: R \to \text{End}(k^m)$, and $A_n$ is the image of this representation. Then $A_n$ is local algebra, e.i. $A_n/J(A_n) = k$.

Proof. This follows from the above lemma 6.1 and fact that any image algebra is basic with semisimple part isomorphic to the sum of $r$ copies of the field $k$: $\oplus_r k$, where $r$ is a number of different eigenvalues of $X$, which was proved in the 2.10. \(\square\)

Now using the definition of quiver for the image algebra given in section 2 and lemma 2.11 we give a complete description of quiver equivalence classes of indecomposable modules.

Corollary 6.4 Quiver corresponding to the indecomposable module has one vertex. The number of loops is one or two, which is a dimension of the vector space $\text{Sp}_k \{ \overline{X} - \lambda I, \overline{Y} \}$, where $\overline{X} = \phi X, \overline{Y} = \phi Y$ for $\phi: A \to A/J^2$.

As another consequence of the proposition 6.2 we can derive an important information on how to glue irreducible modules to get indecomposables. It turned out that it is possible to glue together nontrivially only the copies of the same irreducible module $S_a$.

Corollary 6.5 For arbitrary non-isomorphic irreducible modules $S_a, S_b$,

\[ \text{Ext}_k^1(S_a, S_b) = 0, \text{ if } a \neq b. \]

Proof. Indeed, in corollary 4.2 we derive that irreducible module $S_t$ is one dimensional and given by $X = (a), Y = (0), a \in k$. If $a \neq b$ then for $[M] \in \text{Ext}_k^1(S_a, S_b)$, corresponding $X$ has two different eigenvalues, namely $a$ and $b$. Then by the above lemma $M$ is decomposable and $[M] = 0. \square$

7 Equivalence of some subcategories in mod $R$

Let we denote by mod $R(\lambda)$ the full subcategory in mod $R$ consisting of modules with the unique eigenvalue $\lambda$ of $X$: $\text{mod } R(\lambda) = \{ M \in \text{mod } R | M = M_\lambda(X) \}$. Let us define the functor $F_\lambda$ on mod $R$, which maps a module $M$ to the module $M_\lambda$ with the following new action $rm = \varphi_\lambda(r)m$, where $\varphi_\lambda$ is an automorphism of $R$ defined by $\varphi_\lambda(x) = x + \lambda, \varphi_\lambda(y) = y$. The restriction of $F_\lambda$ to mod $R(\lambda)$ is an equivalence of categories $F_\lambda: \text{mod } R(\lambda) \to \text{mod } R(\mu + \lambda)$ for any $\mu \in k$. In particular, we have an equivalence of the categories mod $R(\lambda)$ and mod $R(0)$.

To use this equivalence of categories it is necessary to know that in most cases (but not in all of them), the eigenvalues of the matrix $X$ are just entries of the main diagonal in the standard shape of the matrix described in the Theorem 4.1, more precisely.
Theorem 7.1 Let in the basis $E$ of the representation vector space, $Y$ is in the Jordan normal form, and Jordan blocks have pairwise different sizes: $n_1, n_2, \ldots, n_k$. Then in the same basis $X$ has the shape (1) with numbers $\lambda_1, \ldots, \lambda_k$ on the diagonals of the main blocks, where $\lambda_j$ are eigenvalues of $X$ (not necessarily different).

Proof. Let we first introduce the denotation for the basis $E$:

$$e^{1,1}, \ldots, e^{1,n_1}, e^{2,1}, \ldots, e^{2,n_2}, \ldots, e^{k,1}, \ldots, e^{k,n_k}.$$

Consider the set $A$ of the matrices (in the same basis) such that $A(j,l),(j,l) = c_j$, $1 \leq j \leq k$, $1 \leq l \leq n_j$ and $A(i,s),(j,l) = 0$ if $n_j < n_i$ and $l > s - n_j$ and if $n_j > n_i$ and $l > s$. One can easily verify that $A$ is an algebra with respect to the matrix multiplication. Let also $D$ be the subalgebra of diagonal matrices in $A$ and $\varphi : A \rightarrow D$ be the natural projection ($\varphi$ acts annihilating the off-diagonal part of a matrix).

Looking at the multiplication in $A$ it is straightforward to see that $\varphi$ is an algebra morphism, that is $\varphi(I) = I$, $\varphi(AB) = \varphi(A)\varphi(B)$ and $\varphi(A+B) = \varphi(A) + \varphi(B)$. It is also easy to check, calculating the powers of the matrix, that if $A \in A$ and $\varphi(A) = 0$ then the matrix $A$ is nilpotent. Since the matrices of the form (1) belong to $A$, it suffices to verify that the eigenvalues of any $A \in A$ coincide with the eigenvalues of $\varphi(A)$.

First, suppose that $\lambda$ is not an eigenvalue of $A$. That is the matrix $A - \lambda I$ is invertible: there exists a matrix $B \in A$ such that $(A - \lambda I)B = I$. Here we use the fact that if a matrix from a subalgebra of the matrix algebra is invertible, then the inverse belongs to the subalgebra. Then $\varphi((A - \lambda I))\varphi(B) = \varphi((A - \lambda I)B) = \varphi(I) = I$. Therefore $\lambda$ is not an eigenvalue of $\varphi(A)$. On the other hand, suppose that $\lambda$ is not an eigenvalue of $\varphi(A)$. Then $\varphi(A) - \lambda I$ is invertible. Clearly

$$A - \lambda I = (\varphi(A) - \lambda I)(I + (\varphi(A) - \lambda I)^{-1}(A - \varphi(A))).$$

Let $B = (\varphi(A) - \lambda I)^{-1}(A - \varphi(A))$. Since $\varphi$ is a projection, we have that

$$\varphi(B) = \varphi((\varphi(A) - \lambda I)^{-1})(\varphi(A) - \varphi(A)) = 0.$$

As we have already mentioned this means that the matrix $B$ is nilpotent and therefore $I + B$ is invertible. Hence $A - \lambda I = (\varphi(A) - \lambda I)(I + B)$ is invertible as a product of two invertible matrices. Therefore $\lambda$ is not an eigenvalue of $A$. Thus, eigenvalues of $A$ and $\varphi(A)$ coincide. This completes the proof. $\square$

8 Analogue of the Gerstenhaber theorem for commuting matrices

In this section we intend to prove an analog of the Gerstenhaber-Taussky-Motzkin theorem (see [8], [19], [10]) on the dimension of images of representations of two generated algebra of commutative polynomials $k[x,y]$. This
Theorem 8.1 Let $\rho_n : R \rightarrow M_n(k)$ be an arbitrary $n$-dimensional representation of $R = k\langle x, y \mid xy - yx = x^2 \rangle$ and $A_n = \rho_n(R)$ be the image algebra. Then the dimension of $A_n$ does not exceed $\frac{n(n+2)}{4}$ for even $n$ and $\frac{(n+1)^2}{4}$ for odd $n$.

This estimate is optimal and attained for the image algebra corresponding to full-block $Y$.

We divide the proof in two lemmas. Start with the second statement of the theorem, that is calculation of the dimension of image algebras in full-block case.

Lemma 8.2 Let $X, Y \in M_n(k)$ be matrices of the size $n$ over the field $k$, satisfying the relation $XY - YX = Y^2$ and $Y$ has as a Jordan normal form one full block. Denote by $A$ the algebra generated by $X$ and $Y$. Then for odd $n = 2m + 1$, $\dim A = \frac{(m+1)^2}{4}$ and for even $n = 2m$, $\dim A = \frac{n(n+2)}{4}$.

Proof. In the Lemma 5.2 we already have computed the matrices, which are images of monomials $y^k x^m$ under the representation $\varepsilon : (x, y) \mapsto (X^0, J_n)$. Due to the fact that any representation $\rho$ with full block $Y$ could be obtained from $\varepsilon$ by composition with the $R$-automorphism $\varphi : x \mapsto x + a, y \mapsto y$, where $[a, y] = 0$, it is enough to calculate the dimensions of images for $\varepsilon$.

Let we recall how matrices $\varepsilon(y^k x^m)$ look like and calculate here the dimension of their linear span.

The matrix $\varepsilon(y^{l-r} x^r)$ on the $l$-th upper diagonal has a vector $(p(0), p(1), \ldots)$, where

$$p(j) = (l - r + j) \ldots (l + j - 1) = \prod_{i=1}^{r} (l + j - r + i)$$

and zeros elsewhere. In the $j$-th place of the $l$-th diagonal we have a value of a polynomial of degree exactly $r$. Those diagonals which have number less then the number of elements in it give the impact to the dimension equal to the dimension of the space of polynomials of corresponding degree. When the diagonals become shorter (the number of elements less then the number of the diagonal) then the impact to the dimension of this diagonal equals to the number of the elements in it. Thus, if $n = 2m + 1$, $\dim A = 1 + \cdots + m + (m + 1) = \frac{(m+1)^2}{4}$. When $n = 2m$, we have $\dim A = 1 + \cdots + m + m + \cdots + 1 = m(m + 1) = \frac{n(n+1)}{4}$. □
8.1 Maximality of the dimension in the full-block case

We know now that any representation of $\mathbb{R}$, which is isomorphic to one with $Y$ in full-block Jordan normal form gives us as an image the same algebra described in lemma 8.2 as a certain set of matrices, of dimension $\frac{n(n+2)}{4}$ for even $n$ and $\frac{(n+1)^2}{4}$ for odd $n$. We intend to prove that this dimension is maximal among dimensions of all image algebras for arbitrary representation, that is the first part of the theorem 8.1.

We start with the proof that this dimension is an upper bound for any image algebra of indecomposable representation.

The simple preliminary fact we will need is the following.

**Lemma 8.3** Matrices $X$ and $Y$ satisfying the relation $XY - YX = Y^2$ can be by simultaneous conjugation brought to a triangular form.

**Proof.** Using the defining relation and the fact that $Y$ is nilpotent we can see that any eigenspace of $Y$ is invariant under $X$. Hence $X$ and $Y$ has joint eigenvector $v$. Then we consider quotient representation on the space $V/\{v\}$ which has the same property. Continuation of this process supply us with the basis where both $X$ and $Y$ are triangular. □

**Lemma 8.4** Let $\rho_n : R \to M_n(k)$ be an indecomposable $n$-dimensional representation of $R$ and $A_n = \rho_n(R)$ be the image algebra. Then the dimension of $A_n$ does not exceed $\frac{n(n+2)}{4}$ for even $n$ and $\frac{(n+1)^2}{4}$ for odd $n$.

**Proof.** The algebra $A_n = \{\sum \alpha_{k,m} Y^k X^m \}$ consists now of triangular matrices. Let we present the linear space $UT_n$ of upper triangular $n \times n$ matrices as the direct sum of two subspaces $UT_n = L_1 \oplus L_2$, where $L_1$ consists of matrices with zeros on upper diagonals with numbers $l, \ldots, n$ and $L_2$ consists of matrices with zeros on upper diagonals with numbers $1, \ldots, l - 1$, where $l = (n + 1)/2$ for odd $n$ and $l = n/2 + 1$ for even $n$. Let $P_j$, $j = 1, 2$ be the linear projection in $UT_n$ onto $L_j$ along $L_{3-j}$. Since $A_n$ is a linear subspace of $UT_n$, we have that $A_n \subset M_1 + M_2$, where $M_j = P_j(A_n)$. Therefore $\dim A_n \leq \dim M_1 + \dim M_2$. The dimension of $M_1$ clearly does not exceed the dimension of the linear span of those matrices $Y^k X^m$, which do not belong to $L_2$. Thus,

$$\dim M_1 \leq \dim (Y^k X^m | k + m < l - 1)_k.$$  

Here we suppose that $X$ (as well as $Y$) is nilpotent. We can do this because the module is indecomposable. Indeed, the lemma 6.1 says that for an indecomposable module $X$ has a unique eigenvalue. This implies that any indecomposable representation is autoequivalent to one with nilpotent $X$ and $Y$ due to the automorphism of $R$ defined by $\varphi_\lambda(x) = x - \lambda$, $\varphi_\lambda(y) = y$. Since autoequivalent representations has the same image algebras we can suppose that $X$ is nilpotent.
Thus the dimension of $M_1$ does not exceed the number of the pairs $(k, m)$ of non-negative integers such that $k + m < l - 1$, which is equal to $1 + \cdots + (l - 1)$. On the other hand $\dim M_2 \leq \dim L_2$ and the dimension of $L_2$ does not exceed the total number of entries in the non-zero diagonals.

$$\dim M_2 \leq 1 + \cdots + (n - l + 1).$$

Taking into account that $\dim A_n \leq \dim M_1 + \dim M_2$, we have

$$\dim A_n \leq 1 + \cdots + (l - 1) + 1 + \cdots + (n - l + 1).$$

The latter sum equals $\frac{n(n+2)}{4}$ for even $n$ and $\frac{(n+1)^2}{4}$ for odd $n$. \Box

After we have proved the estimation for the indecomposable modules, it is easy to see that the same estimate holds for arbitrary module, since the function $n^2$ is convex.

On the other hand as it was shown in the Lemma 8.2 this estimate is attained on the algebra $A_n = \varepsilon(\mathcal{R})$ in the case of a full-block $Y$. This completes the proof of the theorem 8.1.

9 Parametrizable families of representations

Here we suppose that $k = \mathbb{C}$. Let we consider the variety of $R$-module structures on $k^n$ and denote it by $\text{mod}(R, n)$. Such structures are in 1-1 correspondence to a $k$-algebra homomorphisms $R \to M_n(k)$ ($n$-dimensional representations), or equivalently to a pair of matrices $(X, Y)$, $X, Y \in M_n(k)$, satisfying the relation $XY - YX = Y^2$. The group $GL_n(k)$ acts on $\text{mod}(R, n)$ by simultaneous conjugation and orbits of this action are exactly the isomorphism classes of $n$-dimensional $R$-modules. Denote this orbit of a module $M$ or of a pair of matrices $(X, Y)$ as $O(M)$ or $O(X, Y)$ respectively. Consider also the following stratification. Let $\mathcal{U}_P$ be the set of all pairs $(X, Y)$ satisfying the relation, where $Y$ has a fixed Jordan form. Here $\mathcal{P}$ stands for the partition of $n$, which defines the Jordan form of $Y$. Clearly $\mathcal{U}_P$ is a union of all orbits where $Y$ has a Jordan form defined by partition $\mathcal{P}$:

$$\mathcal{U}_P = \bigcup_{Y \text{ with Jordan form defined by the partition } \mathcal{P}} O(X, Y).$$

We will write $\mathcal{U}_{(n)}$ for the stratum corresponding to $Y$ with the full Jordan block: $\mathcal{P} = (n)$.

Another action involved here is an action of the subgroup of $GL_n$ on those pairs $(X, Y)$, where $Y = J_P$ is in fixed Jordan form. Denote this space by $W_P$. The subgroup which acts there is clearly the centralizer of the given Jordan matrix: $Z(J_P)$. Orbits of the action of $Z(J_P)$ on the space $W_P$ are just parts of orbits above: $O_P(X) = O(X, Y) \cap W_P$.

We suggest here to consider in stead of action of $GL_n$ on the whole space an action of centralizer $Z(J_P)$ on the smaller space $W_P$. While the group which acts is not reductive any more and has a big unipotent part, we act just on the space of matrices and some information easier to get in this setting. It then could be (partially) lifted because of 1-1 correspondence of
orbits. More precisely, it could be lifted in sense of parametrization, but if we consider, for example, degeneration of orbits situation may changes after restriction of them.

In this section we will give a parametrization (by two parameters) of the family $M_n$ of representations defined by $\text{rk} Y = n - 1$. What we actually doing here, we obtain this parametrization for $W_{(n)}$. Due to 1-1 correspondence between the orbits we then have a parametrization of $M_n$.

Let we restrict the orbits even a little further, considering the action of the group $G = Z(J_P) \cap SL_n$, where the 1-1 correspondence with the initial orbits will be clearly preserved. In the case $P = (n)$ the group $G$ can be presented as follows:

$$G = \{ I + \alpha Y + \alpha_2 Y^2 + \ldots + \alpha_{n-1} Y^{n-1} \},$$

due to our description of the centralizer of $Y$ in section 4. This group acts on the affine space of the dimension $n$:

$$W_{(n)} = \{ \lambda I + X^0 + c_1 Y + c_2 Y^2 + \ldots + c_{n-1} Y^{n-1} \}$$

here $\lambda$ is the eigenvalue of $X$ and $X^0$ is the matrix defined in section 4 with the second diagonal $[0, 1, \ldots, (n - 1)]$ and zeros elsewhere.

Let we fix first the eigenvalue: $\lambda = 0$, we get then the space of dimension $n - 1$:

$$W'_{(n)} = \{ X^0 + c_1 Y + c_2 Y^2 + \ldots + c_{n-1} Y^{n-1} \}.$$ 

We intend to calculate now the dimension of the orbit $O_{(n)}(X, G)$ of $X$ with fixed eigenvalue $\lambda = 0$ under $G$ – action.

Consider the map $\varphi : G \rightarrow W'_{(n)}$ defined by this action: $\varphi(C) = CXC^{-1}$, then $\text{Im} \varphi = O_{(n)}(X, G)$. We are going to calculate the rank of Jacobian of this map. We will see that it is constant on $G$ and equals to $n - 2$. This tells us that each orbit $O_{(n)}(X, G)$ is an $n - 2$ dimensional manifold and hence there couldn’t be more then 2 parameters involved in parametrization of orbits.

### 9.1 Calculation of the rank of Jacobian

**Theorem 9.1** Let $G$ be an intersection of $SL_n$ with the centralizer of $Y$. Consider the action of this group on the affine space $W'_Y = \{ X^0 + c_1 Y + c_2 Y^2 + \ldots + c_{n-1} Y^{n-1} \}$ by conjugation. Then the rank of the Jacobian of the map $\varphi : G \rightarrow W'_Y$ is equal to $n - 2$ in any point $C \in G$.

**Proof.** Consider $d\varphi(C)(\Delta) = (C + \Delta)^{-1} X(C + \Delta) - C^{-1} XC$, where

$$C = I + \alpha_1 Y + \alpha_2 Y^2 + \ldots + \alpha_{n-1} Y^{n-1},$$

$$X = X^0 + c_1 Y + c_2 Y^2 + \ldots + c_{n-1} Y^{n-1},$$
\[ \Delta = \beta_1 Y + \beta_2 Y^2 + \ldots + \beta_{n-1} Y^{n-1}. \]

Let we present \((C + \Delta)^{-1}\) in the following way:

\[ (C + \Delta)^{-1} = (I + \Delta C^{-1})^{-1} C^{-1} = \]

\[ (I - \Delta C^{-1} + \text{lower order terms on}\Delta) C^{-1}. \]

Then

\[ (C + \Delta)^{-1} X(C + \Delta) - C^{-1} X C = \]

\[ (I - \Delta C^{-1} + \text{lower order terms on}\Delta) C^{-1} X(C + \Delta) - C^{-1} X C = \]

\[ -\Delta C^{-2} X C + C^{-1} X \Delta + \text{lower order terms on}\Delta = \]

\[ (-\Delta C^{-1} \cdot C^{-1} X + C^{-1} X \cdot \Delta C^{-1}) C + \text{lower order terms on}\Delta. \]

Denote \(\tilde{\Delta} := \Delta C^{-1}\) and \(\tilde{X} := C^{-1} X\). Obviously multiplication by \(C\) preserves the rank and rank of linear map \(d\varphi(C)(\Delta)\) is equal to the rank of the map \(T(\tilde{\Delta}) = [\tilde{X}, \tilde{\Delta}]\).

Here again \(\tilde{\Delta}\) has a form

\[ \tilde{\Delta} = \gamma_1 Y + \gamma_2 Y^2 + \ldots + \gamma_{n-1} Y^{n-1}. \]

Let us compute commutator of \(\tilde{X}\) with \(Y^k\). Taking into account that \(C^{-1}\) is a polynomial on \(Y\), hence commute with \(Y^k\) and also the relation in algebra \(\tilde{R}\): \(XY^k - Y^k X = kY^{k+1}\). We get \(\tilde{X} Y - Y \tilde{X} = C^{-1} X Y^k - Y^k C^{-1} X = C^{-1} (X Y^k - Y^k X) = C^{-1} kY^{k+1}\). Hence

\[ \tilde{X} p(Y) - p(Y) \tilde{X} = C^{-1} Y^2 p'(Y) \]

for arbitrary polynomial \(p\). Applying this for the polynomial \(\tilde{\Delta}\) we get

\[ T(\tilde{\Delta}) = [\tilde{X}, \tilde{\Delta}] = \sum_{k=1}^{n-2} \gamma_k k C^{-1} Y^{k+1}, \]

hence this linear map has rank \(n - 2\). \(\square\)

From the theorem 9.1 we could deduce the statement concerning parametrization of isoclasses of modules in the family \(M_n\).

We mean by *parametrization* (by \(m\) parameters) the existence of \(m\) smooth algebraically independent functions which are constant on the orbits and separate them.
Corollary 9.2 Let $\mathcal{U}_n$ be the stratum as above. Then the set of isomorphism classes of indecomposable modules from $\mathcal{U}_n$ could be parameterized by at most two parameters.

Proof. Directly from the theorem 9.1 applying the theorem on locally flat map [6] to $\varphi : G \rightarrow W_{(n)}'$ we have that $\text{Im} \varphi = O_{(n)}(X, G)$ is an $n-2$ dimensional manifold. We have to mention here that this is due to the fact that the image has no selfintersections. This is the case since the preimage of any point $P$ is connected (it is formed just by the solutions of the equation $CX = PC$ for $C \in G$). Hence we can parametrize these orbits lying in the space $W_{(n)}$ of dimension $n$ by at most two parameters. Due to 1-1 correspondence to the whole orbits $O(X, Y)$ the latter have the same property.\[\square\]

Proposition 9.3 Parameters $\mu$ and $\lambda$ are invariant under the action of $G$ on the set of matrices \[
\begin{pmatrix}
\lambda & \mu + 1 & \mu + 2 & \cdots & 0 \\
0 & \lambda & \mu + 3 & \cdots & 0 \\
0 & 0 & \lambda & \mu + n - 1 & 0 \\
\end{pmatrix}.
\]

Proof. Direct calculation of $ZMZ^{-1}$ for $Z \in G$ as described above shows that elements in first two diagonals of $M$ will be preserved. \[\square\]

Hence from the corollary 9.2 and proposition 9.3 we have the following classification result for the family $\mathcal{M}_n$ of representations with full Jordan block $Y$, or equivalently with the condition $n - \text{rk} Y = 1$.

Theorem 9.4 Let $P_{\lambda, \mu}$ denotes the pair $(X_{\lambda, \mu}, Y)$, where
\[
X_{\lambda, \mu} = \begin{pmatrix}
\lambda & \mu + 1 & \mu + 2 & \cdots & 0 \\
0 & \lambda & \mu + 3 & \cdots & 0 \\
0 & 0 & \lambda & \mu + n - 1 & 0 \\
\end{pmatrix}, \quad Y = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

Every pair $(X, Y) \in \mathcal{M}_n$ is conjugate to $P_{\lambda, \mu}$ for some $\lambda, \mu$. No two pairs $P_{\lambda, \mu}$ with different $(\lambda, \mu)$ are conjugate.

Let we mention that number of parameters does not depends of $n$ in this case.

9.2 Some examples of tame strata (up to auto-equivalence)

We collect (quite rare) examples of tame strata in the suggested above stratification related to the Jordan normal form of $Y$. We present here tameness results for the representation type of families of reps lying in the stratum $\mathcal{U}_{(n-1,1)}$ with respect to auto-equivalence relation on modules. It was defined in section 5 and consists of gluing orbits which could be obtained one from another using automorphism of the initial algebra.
Theorem 9.5 The subset of all n-dimensional representations corresponding to \( Y \) with full Jordan block, or equivalently defined by the condition \( n - rY = 1 \), has a finite representation type with respect to auto-equivalence relation on modules.

The subset of all n-dimensional representations corresponding to \( Y \) with the Jordan structure \( \mathcal{P} = (n - 1, 1) \) is tame, that is parametrizable by one parameter, with respect to auto-equivalence relation.

Proof. The proof analogues to the proof of the theorem 9.4. We present here pictures showing how \( X \) and \( Y \) act on basis and where parameters appear:

\[ \mathcal{P} = (n) \]

\[ \begin{array}{cccccccc}
  \bullet & x & \bullet & (x,2) & \bullet & \cdots & \bullet & (x,n-1) \\
  e_1 & y & e_2 & y & e_3 & y & \cdots & e_{n-1} & y & e_n \\
\end{array} \]

\[ \mathcal{P} = (n - 1, 1) \]

\[ \begin{array}{cccccccc}
  \bullet & x & \bullet & (x,2) & \bullet & \cdots & \bullet & (x,n-2) & (x,a) \\
  e_1 & y & e_2 & y & e_3 & y & \cdots & e_{n-2} & y & e_{n-1} & y & e_n \\
\end{array} \]

\[ (x,a^{-1}) \]

\[ \square \]

10 The case of one block and Ringel’s classification of complete local algebras

As we have shown above the set of orbits corresponding to the full-block Jordan structure of \( Y \) in the variety of n-dimensional modules could be parametrized by two parameters. Therefore this family \( \mathcal{M}_n \) of representations defined in section 2 is wild. Nevertheless we intend to prove here that all representations from \( \mathcal{M}_n \) have only one finite dimensional algebra (for any dimension \( n \)) as their image.

We shall show the place of these algebras in Ringel’s classification of complete local algebras [22] by calculating their defining relations. Let we remind that as we have seen in the Corollary 6.3 any indecomposable representation has a local algebra as an image, particularly, representations with full-block \( Y \) do. We are going to prove here that for \( n \leq 4 \) all image algebras \( A_n \) are tame and for \( n \geq 5 \) they are wild.

Theorem 10.1 Let \( \rho_n : R \rightarrow M_n(k) \) be a finite dimensional representation of \( R \), where \( Y = \rho(y) \) has a full-block Jordan structure. Then the image algebra \( A_n = \rho_n(R) \) does not depend on the choice of \( \rho_n \).
Proof. In order to calculate the linear basis of $A_n$ it suffices to find matrices $\varepsilon_n(y^n x^m)$, which are images of normal monomials $y^n x^m$ under the representation $\varepsilon_n$ defined in section 5. It is enough to consider $\varepsilon_n$ since as we have shown in section 5, any $\rho_n$ is auto-equivalent to $\varepsilon_n$ and the images of auto-equivalent representations coincide. These matrices were calculated in the lemma 5.2. Namely, the matrix $\varepsilon_n(y^n x^m)$ has the vector $p(o), p(1), ..., p(j), ...$ of values of polynomial $p(j) = (k + j) ... (k + m + j - 1) = \prod_{i=1}^{m}(k + j + i)$ in the diagonal number $k + m$ and zeros elsewhere. The linear span of these matrices gives us the desired image algebra. \(\square\)

Recall that a $k$-algebra $A$ is called local if $A = k \oplus \text{Jac}(A)$, where $\text{Jac}(A)$ is the Jacobson radical of $A$. One can also consider the completion of $A$: $\overline{A} = \lim A/(\text{Jac}(A))^n$. An algebra $A$ is called complete if $A = \overline{A}$.

It was shown in section 4 that in the case of full-block $Y$, $X$ has only one eigenvalue. We also have proved in Theorem 2.10 that for all image algebras their semisimple part $A/\text{Jac}(A)$ is the direct sum of $r$ copies of $k$, $r$ being the number of different eigenvalues of $X$. Hence in the full-block case the image algebra is local. It is also complete, because $\text{Jac}(A)^N = 0$ for $N$ large enough. Indeed, we can use here the Corollary 2.7 which describe the radical, or observe directly that since $A = k \oplus \text{Jac}(A)$ and $A$ consists of polynomials on $X^0$ and $J_n$ (as an image of one of representations $\varepsilon: (x, y) \mapsto (X^0, J_n)$), then $\text{Jac}(A)$ consists of those polynomials which have no constant term. Since the matrices $J_n$ and $X^0$ are nilpotent of degree $n$ and $n - 1$ respectively, $\text{Jac}(A)^{2n} = 0$.

**Theorem 10.2** The image algebra $A_n$ of a representation $\rho_n \in \mathcal{M}_n$ is wild for any $n \geq 5$. It has a quotient isomorphic to the wild algebra given by relations $y^2, yx - xy, x^2 y, x^3$ from the Ringel’s list of minimal wild local complete algebras. The image algebras $A_1, A_2$ and $A_3$ are tame.

Proof. We intend to show that for $n$ big enough, the algebra $A_n$ has a quotient isomorphic to the algebra $W = (x, y | y^2 = yx - xy = x^2 y = x^3 = 0)$, which is number c) in the Ringel’s list of minimal wild local complete algebras [22].

The algebra $W$ is 5-dimensional. Let us consider the ideal $J$ in $A_n$ generated by the relations above on the image matrices $X$ and $Y$. This ideal has obviously codimension not exceeding 5. We intend to show that $J$ has codimension exactly 5 and therefore $W$ should be isomorphic to $A/J$.

Let us look at the ideal $J$, which is generated by $\{y^2, X^2Y, X^3, XY - YX\}$. First, since $XY - YX = Y^2$, $J$ is generated by $\{Y^2, X^2Y, X^3\}$. It is easy to see that $Y^2$ has zeros on first two diagonals and the vector $1 = (1, \ldots, 1)$ on the third one, $X^2Y, X^3$ have zeros on the first three diagonals. An arbitrary element of $A_n$ has the constant vector $c1$ for some $c \in k$ on the main diagonal. Hence we see that a general element of the ideal $J$ has zeros on the first two diagonals and the constant sequence $c1$ on the third one. Taking into account that $A_n$ comprises the upper triangular matrices that have values of a polynomial of degree at most $m$ on $m$-th
diagonal (lemma 5.2), we see that the main diagonal gives an impact of 1 to the codimension of $J$, the first diagonal gives an impact of 2 to the codimension of $J$ if the length of this diagonal is at least 2 (that is $n \geq 3$) and the second diagonal, — an impact of 2, if the length of this diagonal is at least 3 (that is $n \geq 5$). Thus, $\dim A/J \geq 5$ if $n \geq 5$. This completes the proof in the case $n \geq 5$.

Tameness of $A_1$ and $A_2$ is obvious. For $A_3$ the statement follows from the dimension reason: $\dim A_3 = 4$, it is less than the dimensions of all 2-generated algebras from the Ringel’s list of minimal wild algebras. Since his theorem (theorem 1.4 in [22]) states that any local complete algebra is either tame or has a quotient from the list, $A_3$ can not be wild by dimension reasons. Hence $A_3$ is tame.

Let us consider now the case $n = 4$.

**Theorem 10.3** Let $\rho_4 \in \mathcal{M}_4$ be a four dimensional representation of the algebra $R$. Then the image algebra $A_4 = \rho_4(R)$ is given by the relations $k\langle x, y | x^2 = -2xy, xy = yx + y^2, x^3 = 0 \rangle$ and is tame.

**Proof.** We intend to show that no one of the algebras from the Ringel’s list of minimal wild algebras can be obtained as a quotient of $A_4$. After that using the Ringel’s theorem, we will be able to conclude that it is tame. Suppose that there exists an ideal $I$ of $A_4$ such that $A_4/I$ is isomorphic to $W_j$ for some $j = 1, 2, 3, 4$, where

- $W_1 = k\langle u, v | u^2, uv - \mu vu (\mu \neq 0), v^2u, v^3 \rangle$,
- $W_2 = k\langle u, v | u^2, uv, v^2u, v^3 \rangle$,
- $W_3 = k\langle u, v | u^2, vu, uv^2, v^3 \rangle$,
- $W_4 = k\langle u, v | u^2 - v^2, vu \rangle$.

Since all $W_j$ are 5-dimensional and $A_4$ is 6-dimensional, the ideal $I$ should be one-dimensional. Due to our knowledge on the matrix structure of the algebra $A_4$, we can see that there is only one one-dimensional ideal $I_4$ in $A_4$ and that $I_4$ consists of the matrices with at most one non-zero entry being in the upper right corner of the matrix:

$$I_4 = \left\{ \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

After factorization by this ideal we get a 5-dimensional algebra given by relations

$$\overline{A}_4 = A_4/I = k\langle x, y | x^2 = -2xy, xy = yx + y^2, x^3 = 0, y^3 = 0 \rangle.$$

The question now is whether this algebra is isomorphic to one of the algebras from the above list. Suppose that there exists an isomorphism $\varphi_j : W_j \rightarrow \overline{A}_4$ for some $j \in \{1, 2, 3, 4\}$. Denote $\varphi_j(u) = f_j$ and $\varphi_j(v) = g_j$.  

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First, let us mention that \( f_j \) and \( g_j \) have zero free terms: 
\[
f_j(0) = g_j(0) = 0
\]
because the equalities \( \varphi_j(u^2) = f_j^2 = 0 \) and \( \varphi_j(v^3) = g_j^3 \) imply \( (f_j^{(0)})^2 = (g_j^{(0)})^3 = 0 \) and therefore \( f_j^{(0)} = g_j^{(0)} = 0 \) if \( j = 1, 2, 3 \) and the equalities \( \varphi_4(u^2 - v^2) = f_4^2 - g_4^2 = 0 \) and \( \varphi_4(uv) = f_4 g_4 = 0 \) imply \( (f_4^{(0)})^2 = (g_4^{(0)})^2 \) and \( f_4^{(0)} g_4^{(0)} = 0 \) and therefore \( f_4^{(0)} = g_4^{(0)} = 0 \).

The second observation is that the terms of degree 3 and more are zero in \( A_4 \). Therefore we can present the polynomials \( f_j \) and \( g_j \) as the sum of their linear and quadratic (on \( x \) and \( y \)) parts. So, let
\[
f_j = f_j^{(1)} + f_j^{(2)} \quad \text{and} \quad g_j = g_j^{(1)} + g_j^{(2)},
\]
where
\[
f_j^{(1)} = ax + by, \quad g_j^{(1)} = \alpha x + \beta y, \quad f_j^{(2)} = cyx + dy^2, \quad g_j^{(2)} = \gamma yx + \delta y^2.
\]
In order to get entire linear part of the algebra \( A_4 \) in the range of \( \varphi_j \) we need to have
\[
\det \begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} \neq 0. \tag{3}
\]

For any \( j = 1, 2, 3, 4 \) we are going to obtain a contradiction of the last condition with the equations on \( a, b, \alpha, \beta \) coming from the relations of the algebra \( W_j \).

For instance, consider the case \( j = 2 \). From \( 0 = u^2 = f_j^2 = (f_j^{(1)})^2 = 2(ab - a^2)y + (b^2 + ab - 2a^2)y^2 \) we get \( 2a(b - a) = 0 \) and \( b^2 + ab - 2a^2 = 0 \). The first equation gives us that either \( a = 0 \) or \( a = b \). In the case \( a = 0 \) the second equation implies \( b = 0 \) and the equality \( a = b = 0 \) already contradicts (3). Another solution is \( a = b \neq 0 \). From \( 0 = uv = f_j g_j = f_j^{(1)} g_j^{(1)} = (ax + by)(\alpha x + \beta y) \), substituting \( a = b \), we get \( 0 = a(x + y) (\alpha x + \beta y) = a(\beta - \alpha)(yx + 2y^2) \). Hence \( \beta = \alpha \), which together with the equality \( a = b \) contradicts (3).

In the other three cases one can get a contradiction with (3) along the same lines, which completes the proof. \( \square \)

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