Traffic Models for Dynamic System Optimal Assignment

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Dynamic Traffic Assignment (DTA)

- Travel choice Models
  - Dynamic User Equilibrium (DUE)
  - Dynamic System Optimal (DSO)

- Traffic Models
  - Modeling traffic dynamics

Travel Choice Models

- Dynamic User Equilibrium (DUE)
  - Deterministic DUE
    - For each origin-destination (O-D) pair, the travel costs *experienced* by each traveller, no matter which travel routes or departure times they choose are equal and minimal.
  - Stochastic DUE
    - the travel costs *preceived* by each traveller are equal and minimal.

- Dynamic System Optimal (DSO)
  - travellers cooperate in making their choices for the overall benefit of the system instead of their own individual benefits

Traffic Models - Desirable properties for DTA problem

- Positivity $e \geq 0 \Rightarrow x \geq 0, g \geq 0$
- First-in-first-out $\dot{\tau}(s) \geq 0$
- Flow conservation $x(s) = E(s) - G(s)$ or $\dot{x}(s) = e(s) - g(s)$
- Flow propagation
  - Flow conservation and FIFO: $E(s) = G[\tau(s)]$
  - Differentiate with s gives $e(s) = g[\tau(s)]\,\dot{\tau}(s)$
- Causality

$$\tau(\bar{s}) = \Phi[e(s)] \text{ and } g[\tau(\bar{s})] = \Theta[e(s)], \text{ where } s \leq \bar{s}$$

UCL

Overview of DSO

- Problem description
  - Minimize the total system cost of a network

$$\min_{\mathbf{e}} Z = \sum_{\forall o \in O} \sum_{\forall d \in D} \sum_{\forall p \in P_{od}} \int_0^T \left\{ h_o(s) + [\tau_p(s) - s] + f_d[\tau_p(s)] \right\} \cdot e_p(s) ds$$

s.t. constraints of flow positivity and conservation

  - Note: Adding constraints for FIFO makes the optimization problem non-convex (Carey, 1992)
- to be analyzed using control theory

DSO - Control Theoretical Formulation

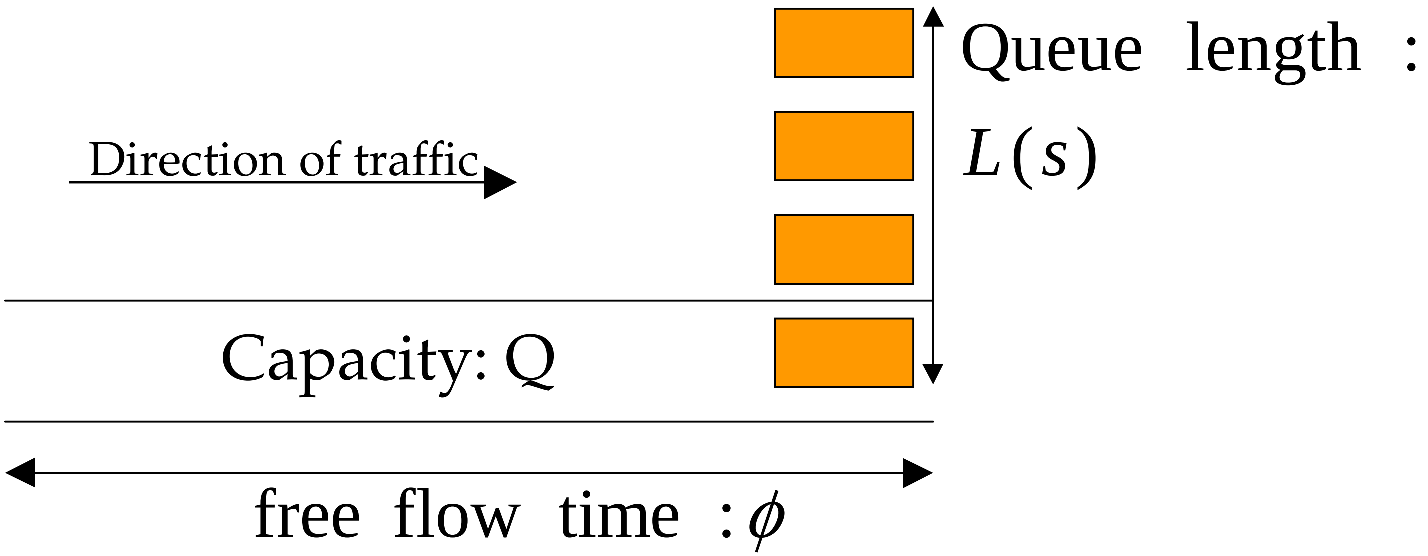
- An analytical method for analysing and solving dynamic optimisation problems
- Variables classified into two categories:
  - control variable ($e(s)$)
  - state variables (other flow quantities, e.g. x(s), g(s))
- Solution: an optimal inflow pattern over time
- Evolutions of state variables (i.e. flow quantities) are governed by state equations
- State equations can be derived from properties of traffic flow model
- Optimal solution exists for linear and continuously differentiable state equations (Kamien and Schwartz, 1991)

Traffic Models Investigated in this study

- Deterministic Queue Model
- Outflow Model (Merchant and Nemhauser, 1978)
- Whole-link traffic models
  - Friesz et al. (1993)’s linear travel time model
  - Carey et al. (2003)’s novel travel time model

Deterministic Queue Model

- Very popular traffic model adopted in DTA
- A freely flowing link together with a D/D/1 queue at its downstream end



Deterministic Queue Model

- The time of exit is determined as

$$\tau(s) = s + \phi + L(s+\phi)/Q$$

- The outflow rate is governed by

$$g(s) = \begin{cases} e(s-\phi) & \left(L(s)=0,\ e(s-\phi)<Q\right) \\ Q & \text{otherwise} \end{cases}$$

- The state equation for *L*:

$$\dot{L}(s) = \begin{cases} 0 & \left(L(s)=0,\ e(s-\phi)<Q\right) \\ e(s-\phi)-Q & \text{otherwise} \end{cases}$$

Deterministic Queue Model

- Satisfies all desirable properties as a plausible traffic model
- The state variable is not differentiable at e(s)=Q

=> difficult to be analysed using control theory and cause non-convexity

$$\dot{L}(s) = \begin{cases} 0 & \left(L(s) = 0,\ e(s-\phi) < Q\right) \\ e(s-\phi) - Q & \text{otherwise} \end{cases}$$

Outflow Model

- Introduced by Merchant and Nemhauser (1978)
- State variable: amount of traffic $x(s)$ along the whole link
- Outflow from each link is considered to be a non-decreasing function $g[x(s)]$ of $x(s)$
- State equation written as

$$\dot{x}(s) = e(s) - g[x(s)]$$

Outflow Model

- Has been widely adopted in DSO analysis using control theory
  - (see for example, Friesz et al., 1989; Yang and Huang, 1997; Wie and Tobin, 1998)
- State equation is continuously differentiable and linear in control variable
- Closed form analytical results have been produced

Outflow Model - Implausible traffic behaviour

- Causality violation (Heydecker and Addison, 1998)

$$x[\tau(s)] = E[\tau(s)] - G[\tau(s)] = E[\tau(s)] - E(s)$$

$$\Rightarrow x[\tau(s)] \text{ and hence } g[\tau(s)] \text{ depends on } e(t), t \in [s, \tau(s)]$$

- FIFO cannot be gauranteed in multi-commodity networks (Astarita, 1996)
- Implausible flow propagation (Astarita, 1996)
  - e.g. last traveller will experience infinite long travel time!

Linear Travel Time Model

- Proposed by Friesz et al. (1993)

$$\tau(s) = s + \phi + x(s)/Q$$

- State equation

$$\dot{x}(s) = e(s) - g(s)$$

- Eliminating $g(s)$ from flow propagation gets

$$\dot{x}(s) = e(s) - e[\sigma(s)] \cdot \dot{\sigma}(s)$$

in which

$$g(s) = e[\sigma(s)]\dot{\sigma}(s) \text{ and } \sigma(s) = s - [\phi + x[\sigma(s)]/Q]$$

Linear Travel Time Model

- Hamiltonian function

$$H = \{h(s) + \tau(s) - s + f[\tau(s)]\} \cdot e(s) + \lambda(s) \cdot \{e(s) - e[\sigma(s)] \cdot \dot{\sigma}(s)\}$$

- Costate equation

$$\dot{\lambda}(s) = -\frac{\partial H}{\partial x(s)} = -(1 + f'[\tau(s)])e(s)\frac{\partial \tau(s)}{\partial x(s)} = -(1 + f'[\tau(s)])\frac{e(s)}{Q}$$

- Optimality conditions (Pontryagain et al., 1962)

$$\frac{\partial H}{\partial e^*(s)} \geq 0$$

$$e^*(s) \cdot \left[\frac{\partial H}{\partial e^*(s)}\right] = 0$$

$$e^*(s) \geq 0$$

Linear Travel Time Model

- For positive optimal inflow

$$\frac{\partial H}{\partial e^*(s)} = \{h(s) + \tau(s) - s + f[\tau(s)]\} + \lambda(s) = 0$$

- Differentiate both sides and sub. $\dot{\lambda}(s)$ gets

$$(h'(s) - 1) + (\dot{\tau}^*(s) - e^*(s)/Q)(1 + f'[\tau(s)]) = 0$$

$$\Rightarrow \dot{\tau}^*(s) = \frac{1 - h'(s)}{1 + f'[\tau(s)]} + \frac{e^*(s)}{Q}$$

- Recall that for Friesz's model $\dot{\tau}(s) = [Q + e(s) - g(s)]/Q$

$$\Rightarrow g^*(s) = \left[\frac{h'(s) + f'[\tau(s)]}{1 + f'[\tau(s)]}\right]Q$$

???

Carey's whole-link model

- Proposed by Carey et al. (2003)
- Travel time - a function of $w(s)$

$$w(s) = \beta e(s) + (1-\beta) g\left[\tau(s)\right]$$

- Causality is ensured
- FIFO is ensured for $\beta \in [0,1)$ (Carey et al., 2003)
- State equation

$$\dot{w}(s) = \beta \dot{e}(s) + (1-\beta) \dot{g}[\tau(s)] \dot{\tau}(s)$$

Depends on derivative of e(s) !

Carey's whole-link model

$$\dot{w}(s) = \beta\dot{e}(s) + (1-\beta)\dot{g}[\tau(s)]\dot{\tau}(s)$$

- State equation depends on the first derivative of the control variable
- No straightforward analysis
- Existence of solution cannot be guaranteed

Concluding Remarks

- Overview of requirements for traffic model in DTA problem
- Overview of DSO control theoretical formulation
- Investigate four different traffic models for their applicability in DSO assignment and suitability in control theoretical formulation

Concluding Remarks

- Deterministic queue model does not have a continuously differentiable state variable
- Ouflow models can not represent plausible traffic behaviour
  - e.g. violation of FIFO and causality; implausible flow propagation
- State equation in Carey's model depends on the first order derivative of the control variable
  - need to be further explored

Concluding Remarks

- Linear travel time models suit for analysis using control theory:
  - plausible traffic behaviour
  - continuously differentiable and linear state equation
  - Furthe investigation in linearity and optimal solution