Traffic Models for Dynamic System
Optimal Assignment

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Dynamic Traffic Assignment (DTA)

• Travel choice Models
  – Dynamic User Equilibrium (DUE)
  – Dynamic System Optimal (DSO)

• Traffic Models
  – Modeling traffic dynamics
Travel Choice Models

- **Dynamic User Equilibrium (DUE)**
  - Deterministic DUE
    - For each origin-destination (O-D) pair, the travel costs *experienced* by each traveller, no matter which travel routes or departure times they choose are equal and minimal.
  - Stochastic DUE
    - the travel costs *perceived* by each traveller are equal and minimal.

- **Dynamic System Optimal (DSO)**
  - travellers cooperate in making their choices for the overall benefit of the system instead of their own individual benefits
Traffic Models - Desirable properties for DTA problem

- **Positivity**  \( e \geq 0 \Rightarrow x \geq 0, g \geq 0 \)
- **First-in-first-out**  \( \dot{\tau}(s) \geq 0 \)
- **Flow conservation**  \( x(s) = E(s) - G(s) \) or  \( \dot{x}(s) = e(s) - g(s) \)
- **Flow propagation**
  - Flow conservation and FIFO:  \( E(s) = G[\tau(s)] \)
  - Differentiate with \( s \) gives  \( e(s) = g[\tau(s)] \dot{\tau}(s) \)
- **Causality**
  \[ \tau(\bar{s}) = \Phi[e(s)] \text{ and } g[\tau(\bar{s})] = \Theta[e(s)], \text{ where } s \leq \bar{s} \]
Overview of DSO

• Problem description
  – Minimize the total system cost of a network

\[
\min Z = \sum_{e} \sum_{o \in O} \sum_{d \in D} \sum_{p \in P_{od}} \int_{0}^{T} \left\{ h_{o}(s) + [\tau_{p}(s) - s] + f_{d}[\tau_{p}(s)] \right\} \cdot e_{p}(s) \, ds
\]

s.t. constraints of flow positivity and conservation

  – Note: Adding constraints for FIFO makes the optimization problem non-convex (Carey, 1992)

• to be analyzed using control theory
• An analytical method for analysing and solving dynamic optimisation problems
• Variables classified into two categories:
  – control variable \((e(s))\)
  – state variables (other flow quantities, e.g. \(x(s), g(s)\))
• Solution: an optimal inflow pattern over time
• Evolutions of state variables (i.e. flow quantities) are governed by state equations
• State equations can be derived from properties of traffic flow model
• Optimal solution exists for linear and continuously differentiable state equations (Kamien and Schwartz, 1991)
Traffic Models Investigated in this study

- Deterministic Queue Model
- Outflow Model (Merchant and Nemhauser, 1978)
- Whole-link traffic models
  - Friesz et al. (1993)’s linear travel time model
  - Carey et al. (2003)’s novel travel time model
Deterministic Queue Model

- Very popular traffic model adopted in DTA
- A freely flowing link together with a D/D/1 queue at its downstream end

Direction of traffic

Capacity: Q

Queue length: $L(s)$

Free flow time: $\phi$
The time of exit is determined as
\[ \tau(s) = s + \phi + \frac{L(s + \phi)}{Q} \]

The outflow rate is governed by
\[ g(s) = \begin{cases} 
    e(s - \phi) & (L(s) = 0, \ e(s - \phi) < Q) \\
    Q & \text{otherwise}
\end{cases} \]

The state equation for \( L \):
\[ \dot{L}(s) = \begin{cases} 
    0 & (L(s) = 0, \ e(s - \phi) < Q) \\
    e(s - \phi) - Q & \text{otherwise}
\end{cases} \]
Deterministic Queue Model

- Satisfies all desirable properties as a plausible traffic model
- The state variable is not differentiable at \( e(s) = Q \)
  => difficult to be analysed using control theory
  and cause non-convexity

\[
\dot{L}(s) = \begin{cases} 
0 & (L(s) = 0, e(s - \phi) < Q) \\
 e(s - \phi) - Q & \text{otherwise}
\end{cases}
\]
Outflow Model

- Introduced by Merchant and Nemhauser (1978)
- State variable: amount of traffic $x(s)$ along the whole link
- Outflow from each link is considered to be a non-decreasing function $g[x(s)]$ of $x(s)$
- State equation written as

$$
\dot{x}(s) = e(s) - g[x(s)]
$$
Outflow Model

• Has been widely adopted in DSO analysis using control theory
  – (see for example, Friesz et al., 1989; Yang and Huang, 1997; Wie and Tobin, 1998)
• State equation is continuously differentiable and linear in control variable
• Closed form analytical results have been produced
Outflow Model - Implausible traffic behaviour

• Causality violation (Heydecker and Addison, 1998)
  \[ x[\tau(s)] = E[\tau(s)] - G[\tau(s)] = E[\tau(s)] - E(s) \]
  \[ \Rightarrow x[\tau(s)] \text{ and hence } g[\tau(s)] \text{ depends on } e(t), t \in [s, \tau(s)] \]

• FIFO cannot be guaranteed in multi-commodity networks (Astarita, 1996)

• Implausible flow propagation (Astarita, 1996)
  – e.g. last traveller will experience infinite long travel time!
Linear Travel Time Model

• Proposed by Friesz et al. (1993)

\[ \tau(s) = s + \phi + x(s) / Q \]

• State equation

\[ \dot{x}(s) = e(s) - g(s) \]

• Eliminating \( g(s) \) from flow propagation gets

\[ \dot{x}(s) = e(s) - e[\sigma(s)] \cdot \dot{\sigma}(s) \]

in which

\[ g(s) = e[\sigma(s)] \dot{\sigma}(s) \quad \text{and} \quad \sigma(s) = s - [\phi + x[\sigma(s)] / Q] \]
Linear Travel Time Model

- **Hamiltonian function**

\[ H = \{h(s) + \tau(s) - s + f[\tau(s)]\} \cdot e(s) + \lambda(s) \cdot \{e(s) - e[\sigma(s)] \cdot \dot{\sigma}(s)\} \]

- **Costate equation**

\[ \dot{\lambda}(s) = -\frac{\partial H}{\partial x(s)} = -(1 + f'[\tau(s)])e(s) \frac{\partial \tau(s)}{\partial x(s)} = -(1 + f'[\tau(s)])\frac{e(s)}{Q} \]

- **Optimality conditions (Pontryagin et al., 1962)**

\[ \frac{\partial H}{\partial e^*(s)} \geq 0 \]

\[ e^*(s) \cdot \left[ \frac{\partial H}{\partial e^*(s)} \right] = 0 \]

\[ e^*(s) \geq 0 \]
Linear Travel Time Time Model

- For positive optimal inflow
  \[
  \frac{\partial H}{\partial e^*(s)} = \left\{ h(s) + \tau(s) - s + f[\tau(s)] \right\} + \lambda(s) = 0
  \]
- Differentiate both sides and sub. \( \dot{\lambda}(s) \) gets
  \[
  (h'(s) - 1) + \left( \dot{\tau}^*(s) - e^*(s) / Q \right) \left[ 1 + f'[\tau(s)] \right] = 0
  \]
  \[
  \Rightarrow \dot{\tau}^*(s) = \frac{1 - h'(s)}{1 + f'[\tau(s)]} + \frac{e^*(s)}{Q}
  \]
- Recall that for Friesz’s model \( \dot{\tau}(s) = [Q + e(s) - g(s)] / Q \)
  \[
  \Rightarrow g^*(s) = \left[ \frac{h'(s) + f'[\tau(s)]}{1 + f'[\tau(s)]} \right] Q
  \]
Carey’s whole-link model

- Proposed by Carey et al. (2003)
- Travel time - a function of $w(s)$

$$w(s) = \beta e(s) + (1 - \beta) g[\tau(s)]$$

- Causality is ensured
- FIFO is ensured for $\beta \in [0,1)$ (Carey et al., 2003)
- State equation

$$\dot{w}(s) = \beta \dot{e}(s) + (1 - \beta) \dot{g}[\tau(s)]\dot{\tau}(s)$$

Depends on derivative of $e(s)$!
Carey’s whole-link model

\[ \dot{w}(s) = \beta \dot{e}(s) + (1 - \beta) \dot{g}[\tau(s)] \dot{\tau}(s) \]

- State equation depends on the first derivative of the control variable
- No straightforward analysis
- Existence of solution cannot be guaranteed
Concluding Remarks

• Overview of requirements for traffic model in DTA problem
• Overview of DSO control theoretical formulation
• Investigate four different traffic models for their applicability in DSO assignment and suitability in control theoretical formulation
Concluding Remarks

- Deterministic queue model does not have a continuously differentiable state variable
- Outflow models can not represent plausible traffic behaviour
  - e.g. violation of FIFO and causality; implausible flow propagation
- State equation in Carey’s model depends on the first order derivative of the control variable
  - need to be further explored
• Linear travel time models suit for analysis using control theory:
  – plausible traffic behaviour
  – continuously differentiable and linear state equation
  – Furthe investigation in linearity and optimal solution