The Influence of Structural Modeshape Variability on Limit Cycle Oscillation Behaviour

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Modal analysis is a popular approach used in structural dynamic and aeroelastic problems due to its efficiency. The response of a structure is composed of the sum of orthogonal eigenvectors or modeshapes and corresponding modal frequencies. This paper investigates the importance of modeshapes on the aeroelastic response of the Goland wing subject to structural uncertainties. The wing undergoes limit cycle oscillations (LCO) as a result of the inclusion of polynomial stiffness nonlinearities. The LCO computations are performed using a Harmonic Balance approach for speed, the modal properties of the system are extracted from MSC NASTRAN. Variability in both the wing’s structure and the store centre of gravity location is investigated in two cases:- supercritical and subcritical type LCOs. Results show that the LCO behaviour is only sensitive to change in modeshapes when the nature of the modes are changing significantly.

Nomenclature

English Letters

A_i rationally approximated components of generalised aerodynamic matrix, \( i = 1, \ldots n_l + 2 \)
F, Q vector of Fourier coefficients
\( g, h \) localised cubic/pentic stiffness coefficient
J Newton-Raphson system Jacobian
\( n_l \) number of lag variables in rational approximation
\( n_m \) number of orthogonal modes
\( N_H \) number of harmonics in Fourier expansion
q vector of modal deflections
\( q_{ai} \) decomposed generalised aerodynamic vectors, \( i = 1, \ldots, n_l \)
\( R^n \) Newton-Raphson system residual vector at iteration \( n \)
\( S^n \) Newton-Raphson solution vector at iteration \( n \)
t time
V freestream velocity

Greek and Roman Symbols

\( \beta_i \) term from rational approximation of generalised aerodynamic forces, \( i = 1, \ldots, n_l \)
\( \lambda \) Newton-Raphson relaxation parameter
\( \rho \) air density
\( \Phi \) truncated matrix of eigenvectors
\( \omega \) fundamental solution frequency

Subscripts and Superscripts

(\()_{ai} \) decomposed generalised aerodynamic vectors
(\()_{nl} \) nonlinear force
(\()_\phi \) quantity in modal domain
(\()^z \) translation about z axis dof
(\()^y \) rotation about y axis dof

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I. Introduction

In the aircraft industry, with regards to aeroelasticity in particular, physical testing can be both costly and dangerous. This has driven the continuous development of simulation techniques to aid in the design and certification of aircraft. Modern aircraft are assembled from a large number of irregular and intricate components. The complexity associated with these structures, arising from interactions between components and from the individual parts themselves magnifies the uncertainty in how the structure will behave. Thus predicting the aeroelastic stability of an aircraft should capture the consequences of variability or uncertainty in model parameters, as discussed by Pettit.\(^1\) Marques et al demonstrated the significant impact of structural variability on transonic flutter predictions.\(^2,3\) When nonlinearities are present, the amplitude of oscillations can become limited and limit cycle oscillations are observed. This is a problem of considerable practical interest and is well documented for in-service aircraft.\(^4,5\) When nonlinearities are de-stabilizing (softening) a subcritical limit-cycle exists. As discussed by Stanford and Beran,\(^6\) unstable LCOs can occur below the flutter speed and lead to a hysteretic phenomenon. This type of instability is extremely undesirable because as the flutter speed is reached, the amplitude increases suddenly and significantly, as the speed drops below the flutter point, the LCO will persist.

The presence of nonlinearities, either structural or aerodynamic, poses additional challenges both in terms of complexity and computational resources, these requirements can be exacerbated by the need for probabilistic analysis. Hence, to address this issue, a Harmonic Balance (HB) based method is implemented for the LCO simulation. The HB method can offer over one order of magnitude reduction in computational effort when compared against time domain methods.\(^5,7–9\) An overview of different variations of the Harmonic Balance method, such as high-dimensional, incremental, or elliptic HB methods is given by Dimitriadis.\(^10\)

Modal analysis is employed to reduce the complexity of the model. The effects of structural variability on the LCO behaviour of the Goland wing is of interest in this work. This variability manifests itself in the orthogonal modes that describe the wing’s structural behaviour in the form of modeshapes and modal frequencies. The extraction of the modeshapes accounts for a relatively small proportion of the cost of a deterministic LCO simulation when using the HB method, however when performed many times as in stochastic analysis, this cost can be substantial. The main focus of this work is determining whether propagating the structural uncertainty through the modal frequencies alone is sufficient to capture how the LCO behaviour of the Goland wing varies.

Two types of structural variability are considered in this work:- 1. Variability in the wing, ie. skin thickness, spar thickness and spar cap areas, a total of 7 uncertain parameters are used. 2. Variability in the store attachment via the centre of gravity location. MSC NASTRAN is utilised to compute the modeshapes and frequencies which are fed into the aeroelastic model. Linear aerodynamics are used in the aeroelastic analysis, rationally approximated from AIC matrices also obtained in NASTRAN.

In this work, a description of the modal aeroelastic formulation with regards to the Goland wing is presented. Following this the implementation of the Harmonic Balance method is described. The paper will then assess the importance of modeshape updating on LCO behaviour for two different polynomial nonlinearities resulting in two types of LCO:- a hardening cubic stiffness nonlinearity which induces a supercritical bifurcation and a pentic stiffness nonlinearity with softening cubic component which results in a subcritical bifurcation.

II. Goland Wing Formulation

The Goland wing provides a relatively simple 3D model exhibiting several complex aeroelastic phenomenon that are challenging to engineering prediction methods. The finite element model of the heavy version of the Goland wing is described in\(^11\) and is shown in Figure 1(a). MSC NASTRAN is used to extract the structural and aerodynamic characteristics of the wing, SOL 103 for the normal modes analysis and SOL 145 for the p-k flutter analysis.\(^12,13\) The structural model used in this work includes localised nonlinearities in the attachment stiffness between an external store and the wing. The nonlinearities are in the form of polynomial laws for spring elements in the translational $z$-direction ($K_z$), and the rotational $y$-direction ($K_{ry}$) degrees of freedom which were shown to be the most sensitive in reference.\(^14\) The problem is analysed in the modal domain as it significantly reduces the number of degrees of freedom of the problem.\(^15\) For this
investigation only four structural modes are considered and are shown in Figure 1(b). The system in the modal domain described by a set of modal coordinates, \( q \) consisting of \( n_m \) modes is given as:\(^5\)\(^14\)

\[
\left[ \tilde{M}_\phi \right] \{ \ddot{q} \} + \left[ \tilde{C}_\phi \right] \{ \dot{q} \} + \left[ \tilde{K}_\phi \right] \{ q \} + [\Phi]^T \{ f_{nl} \{ \hat{q} \} \} = \frac{\rho V^2}{2} \sum_{i=1}^{n_l} [A_{i+2}] \{ \dot{q}_{i+1} \} \tag{1}
\]

where \( \tilde{M}_\phi, \tilde{C}_\phi \) and \( \tilde{K}_\phi \) are the aeroelastic system mass, damping and linear stiffness matrices, \( [\Phi]^T \{ f_{nl} \{ \hat{q} \} \} \) is the nonlinear force in the modal domain, shown in Appendix A where \( [\Phi]^T \) is a truncated set of eigenvectors which transforms from the physical to modal space. \( A_\ast \) represents the rationally approximated components of the generalised aerodynamic matrix extracted from NASTRAN and \( \dot{q}_{i+1} \) are augmented terms arising from the Laplace domain treatment of the generalised aerodynamic matrix which have the relationship:

\[
\dot{q}_{i+1} = \dot{q} - \frac{V}{b} \beta_i \dot{q}_{i+1} \tag{2}
\]

The state space equation is constructed by combining eqs. (1) and (2), leading to:

\[
\{ \dot{w} \} + [A_s] \{ w \} + \{ u \} = \{ 0 \} = \{ R \} \tag{3}
\]

where

\[
[A_s] = \begin{bmatrix}
0 & -I & 0 & \ldots & \ldots & 0 \\
\tilde{M}_\phi^{-1} \tilde{K}_\phi & \tilde{M}_\phi^{-1} \tilde{C}_\phi & \tilde{M}_\phi^{-1} \frac{V^2}{2} I & \ldots & \ldots & \tilde{M}_\phi^{-1} \frac{V^2}{2} I \\
0 & A_3 & \frac{V}{b} \beta_1 I & 0 & \ldots & 0 \\
0 & A_4 & 0 & \frac{V}{b} \beta_2 I & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
0 & A_{n_l+2} & 0 & \ldots & 0 & \frac{V}{b} \beta_{n_l} I
\end{bmatrix}^T
\]

\{ u \} = \begin{bmatrix}
[\tilde{M}_\phi]^{-1} \Phi^T \{ f_{nl} \{ \hat{q} \} \} \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}

System (3) describes how the wing behaves in the time domain and is used to validate the HB solver.
III. Harmonic Balance Implementation

For periodic problems such as LCOs, typical time domain methods such as eq. (3) can be inefficient due to the computation of redundant transient information as the LCO evolves. The Harmonic Balance method avoids this expense by computing the periodic state directly. Further details of Harmonic Balance are readily available. The solution of eq. (1) can be represented as Fourier series truncated to \( N_{H} \) harmonics with a fundamental frequency \( \omega \):

\[
q(i) \approx \hat{q}_{0}^i + \sum_{n=1}^{N_{H}} (\hat{q}_{2n-1}^i \cos \omega n t + \hat{q}_{2n}^i \sin \omega n t)
\] (4)

The first and second derivatives of \( x(t) \) with respect to time can be found to be:

\[
\dot{q}(i) \approx \sum_{n=1}^{N_{H}} (-\omega n \hat{q}_{2n-1}^i \sin \omega n t + \omega n \hat{q}_{2n}^i \cos \omega n t)
\] (5)

\[
\ddot{q}(i) \approx \sum_{n=1}^{N_{H}} (-\omega^2 n^2 \hat{q}_{2n-1}^i \sin \omega n t - (\omega n)^2 \hat{q}_{2n}^i \sin \omega n t)
\] (6)

where \( i = 1, ..., n_{m} \). The orthogonality of Fourier series allows the collecting of Fourier coefficients from each harmonic to form independent equations, this is known as harmonic balancing. It is possible to cast the problem in the frequency domain provided every term in the equation can be described by the same set of harmonic to form independent equations, this is known as harmonic balancing. It is possible to cast the problem in the frequency domain provided every term in the equation can be described by the same set of Fourier coefficients. However, here the complexity of the nonlinear force and aerodynamic terms prevent this so a direct substitution of the Fourier expansions is used thus remaining in the time domain. The Fourier series can be represented as the product of a transformation matrix and a vector of Fourier coefficients so eq. (1) can take the form:

\[
\begin{bmatrix}
\tilde{M}_{x} & \tilde{C}_{x} & \tilde{K}_{x}
\end{bmatrix} \{ \Phi \} + \begin{bmatrix}
E_{vel}^{-1} & 0 & 0
\end{bmatrix} \{ \Phi \} + \begin{bmatrix}
E_{def}^{-1}
\end{bmatrix} \{ \Phi \} - \frac{\rho V^2}{2} \sum_{i=1}^{n_{i}} \begin{bmatrix}
A_{i+2}
\end{bmatrix} \begin{bmatrix}
E_{def}^{-1}
\end{bmatrix} \{ \Phi \} = 0
\] (7)

where

\[
\begin{bmatrix}
\tilde{M}_{x} & \tilde{C}_{x} & \tilde{K}_{x}
\end{bmatrix} \{ \Phi \} + \begin{bmatrix}
E_{vel}^{-1} & 0 & 0
\end{bmatrix} \{ \Phi \} + \begin{bmatrix}
E_{def}^{-1}
\end{bmatrix} \{ \Phi \} - \frac{\rho V^2}{2} \sum_{i=1}^{n_{i}} \begin{bmatrix}
A_{i+2}
\end{bmatrix} \begin{bmatrix}
E_{def}^{-1}
\end{bmatrix} \{ \Phi \} = 0
\]

and

\[
\begin{bmatrix}
\hat{f}_{0}^1 \\
\hat{f}_{1}^1 \\
\vdots \\
\hat{f}_{2N_{H}}^1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{f}_{0}^{a_{1},1} \\
\hat{f}_{1}^{a_{1},1} \\
\vdots \\
\hat{f}_{2N_{H}}^{a_{1},1}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\hat{f}_{0}^{a_{m},1} \\
\hat{f}_{1}^{a_{m},1} \\
\vdots \\
\hat{f}_{2N_{H}}^{a_{m},1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{f}_{0}^{a_{m},n} \\
\hat{f}_{1}^{a_{m},n} \\
\vdots \\
\hat{f}_{2N_{H}}^{a_{m},n}
\end{bmatrix}
\]

\[
E_{acc}^{-1}, E_{vel}^{-1} \text{ and } E_{def}^{-1} \text{ are the transformation matrices and are shown in Appendix B. The Fourier coefficients for the nonlinear force, } \{ \Phi \} \text{ and the decomposed generalised aerodynamic vectors, } \{ \Phi_{n} \} \text{ are described using discrete Fourier transforms (DFT) and are evaluated numerically using the trapezoidal rule from expressions shown in Appendix B. Equation (7) is solved simultaneously for } 2N_{H} + 1 \text{ equally spaced time steps across one period with } t_{i} = \frac{2\pi}{2N_{H} + 1}, (i = 0, 1, ..., 2H) \text{ to maintain temporal accuracy. A Newton-Raphson scheme is employed and is given as:

\[
S^{n+1} = S^{n} - \lambda J^{-1} R^{n}
\] (8)

where } S^{n} \text{ is the solution vector at iteration } n, \lambda \text{ is a relaxation parameter for increased stability, } J^{-1} \text{ is the numerically approximated inverse Jacobian of the system and } R^{n} \text{ is the residual of eq. (7) at iteration } n, \text{ ie.}
\[ S^n = \left\{ \begin{array}{c} \hat{q}_0^1 \\
 \omega \\
 \vdots \\
 \hat{q}_{2N_H}^{n_2} \end{array} \right\}, \quad R^n = \left\{ \begin{array}{c} R_0^1 \\
 R_1^1 \\
 \vdots \\
 R_{2N_H}^{n_2} \end{array} \right\} \]

and

\[ J = \left[ \begin{array}{ccc} \frac{\partial R_0^1}{\partial \hat{q}_0^1} & \frac{\partial R_1^1}{\partial \omega} & \cdots & \frac{\partial R_{2N_H}^{n_2}}{\partial \hat{q}_{2N_H}^{n_2}} \\
 \frac{\partial R_0^1}{\partial \omega} & \frac{\partial R_1^1}{\partial \omega} & \cdots & \frac{\partial R_{2N_H}^{n_2}}{\partial \hat{q}_{2N_H}^{n_2}} \\
 \vdots & \vdots & \ddots & \vdots \\
 \frac{\partial R_0^1}{\partial \hat{q}_{2N_H}^{n_2}} & \frac{\partial R_1^1}{\partial \hat{q}_{2N_H}^{n_2}} & \cdots & \frac{\partial R_{2N_H}^{n_2}}{\partial \hat{q}_{2N_H}^{n_2}} \end{array} \right] \]

Note the presence of frequency, \( \omega \) in the solution vector and Jacobian. This enables the problem to be solved when the frequency of the response is not known \textit{a priori}. The frequency replaces the cosine Fourier coefficient of the first harmonic of the first mode which is set to zero, i.e. \( \hat{q}_1^1 = 0 \). This locks the phase of the solution without affecting the amplitude of the response. The Jacobian terms are computed numerically using finite differences:

\[ \frac{\partial R_i}{\partial S_j} \approx \frac{R_i(S, S_j + \delta) - R_i(S)}{\delta} \quad \text{with } i, j = (0, 1, 2, \ldots, n_m(2N_H + 1) - 1) \]

where \( \delta \) is a small perturbation, typically in the order of \( 10^{-3} \).18

**IV. Results**

**IV.A. HB Validation**

For validation of the HB solver, velocity sweeps were performed on the deterministic model and time domain simulations where computed at discrete velocities. Both supercritical and subcritical cases were validated. The supercritical case provides good agreement between the HB and time domain simulations and is shown in Figure 2. Note that the HB appears to be more sensitive to velocity but is adequate for the region of interest.

![Figure 2. Supercritical behaviour](image)

For the subcritical case, the direction of the velocity sweep has an impact on what branch of LCO the solution will lock into as shown in Figure 3. For upward velocity incrementation, the time domain simulations show the sudden onset of LCO when the velocity is increased above the flutter speed. For a descending velocity sweep the LCO will persist below the flutter speed along the top branch. The HB curve is found by running 100 samples with variation in initial amplitude and velocity. It is important to mention that the time domain model cannot capture the region of hysteresis (middle branch) or the 'no
LCO’ condition above the flutter speed as they are unstable solutions. In reality, only solutions along the top branch and bottom branch inside the subcritical region can be achieved. Hence, any solutions on the unstable middle branch computed by the HB method can be assumed to jump upwards onto the upper stable branch.

The variability of the LCO behaviour is assessed using typical Monte-Carlo Analysis where 10,000 Latin hypercube samples are used for all cases. The impact of two types of structural uncertainty is considered over a range of velocities for both supercritical and subcritical cases. A comparison will be made between complete modal updating (ie. modeshapes and frequencies) and modal frequency updating only.

IV.B. Wing Variability

The structural uncertainty is propagated through 7 parameters which define the composition of the wing and are described in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Nominal value (ft)</th>
<th>Range</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper skin thickness</td>
<td>0.0155</td>
<td>±5%</td>
<td>uniform</td>
</tr>
<tr>
<td>Lower skin thickness</td>
<td>0.0155</td>
<td>±5%</td>
<td>uniform</td>
</tr>
<tr>
<td>LE spar thickness</td>
<td>0.0006</td>
<td>±5%</td>
<td>uniform</td>
</tr>
<tr>
<td>TE spar thickness</td>
<td>0.0066</td>
<td>±5%</td>
<td>uniform</td>
</tr>
<tr>
<td>LE spar cap area (ft$^2$)</td>
<td>0.0416</td>
<td>±5%</td>
<td>uniform</td>
</tr>
<tr>
<td>Centre spar cap area (ft$^2$)</td>
<td>0.1496</td>
<td>±5%</td>
<td>uniform</td>
</tr>
<tr>
<td>TE spar cap area (ft$^2$)</td>
<td>0.0416</td>
<td>±5%</td>
<td>uniform</td>
</tr>
</tbody>
</table>

IV.B.1. Supercritical Bifurcations

The LCOs are a result of the inclusion of localised polynomial stiffness nonlinearities in the store attachment as described in reference.\textsuperscript{14} For this case the cubic stiffness coefficients in the rotational $\phi$ axis and translational $z$ dof were given values of $10^{10}$ and $10^{10.9542}$ respectively. The damping matrix, $\tilde{C}_\phi$ is given a damping coefficient of 0.1. Figure 4 shows the distribution of LCO amplitude when the wing is subject to the variability described in Table 1 at 700 ft/s. There is good agreement between full and partial updating suggesting that modeshape updating is not necessary at this velocity.

Figure 5 shows the LCO behaviour of the wing across a range of velocities for full and partial updating. The colours show the density of response points in near proximity where red denotes a high concentration of points and blue indicates a low number of points. In this case a substantial fraction of the points exhibited...
an amplitude of zero (ie. no LCO) so 35% of the samples with the lowest amplitudes were removed to clarify the areas where LCOs are exhibited. The supercritical behaviour is clearly demonstrated by the heat-maps and both cases show excellent agreement. Here neglecting the modeshapes appears to have little impact on the behaviour of the wing.

**Figure 4. Supercritical LCO response, wing variability, 700ft/s**

**IV.B.2. Subcritical Bifurcations**

The subcritical LCOs are achieved by including pentic stiffness terms to the store attachment. The pentic stiffness coefficients in the rotational \( \text{dof} \) about the \( y \) axis and translational \( z \) \( \text{dof} \) were given values of \( 10^{16.5} \) and the cubic terms set to \( 10^{11.5} \) and \( 10^{10} \) respectively. It is important to reiterate that the cubic components of stiffness are now softening (-) while the pentic components are hardening (+). The LCO amplitude distribution for the subcritical case at 655ft/s is shown in Figure 6. As in the supercritical case, the partial updating performs well showing good agreement with the full updating case.

The LCO behaviour over a range of velocities is shown in Figure 7 where the subcritical behaviour is

**Figure 5. Supercritical LCO response, wing variability, top 65% of samples**
captured clearly. It is worth mentioning that the initial tip deflection of the wing was also varied uniformly in all the subcritical cases between 0 and 1.5 in. This enables the simulation to converge to any of the branches on the subcritical curve including the unstable middle branch. Here 35% of the samples were removed for clarity. The neglected samples were all located on the bottom branch (no LCO), most of them at low velocities. The variation of amplitudes on the middle branch is much greater than exhibited on the stable upper branch although more samples reside in this upper branch. The similarities in results exhibited by the full and partial updating has shown that this type of variability on the wing components does not have significant impact on the modeshapes of the structure. The modal frequencies can then be assumed responsible for the change in the wing’s behaviour.

IV.C. Store Centre of Gravity Location Variability

In this case the wing’s structure remains unchanged. The store attachment position is varied in the chord-wise axis. The nominal location of the store mass is 0.25 ft downstream of the leading edge. Here, the centre
of gravity location is considered top move ±0.5 ft from the deterministic value. This will shift the elastic axis of the wing and significantly alter the aeroelastic stability. As before this uncertainty is represented by a uniform distribution.

**IV.C.1. Supercritical Bifurcations**

The nonlinearities remain the same as in the wing variability cases and the deterministic wing parameters are used. Figure 8 shows the distribution of LCO amplitude at 700 ft/s when the wing is subject to variability in the store centre of gravity location. Contrary to the wing variability case it is clear that the exclusion of modeshape updating results in radically different behaviour than that of full updating. Partial updating exhibits a smooth distribution of LCO amplitudes whereas full updating shows an irregular distribution of amplitudes. Both cases contain a large portion of samples which do not produce LCOs due to the increased stability associated some store locations. Note that much large LCO amplitudes are possible when modeshapes are updated. Clearly it would be dangerous to update only the modal frequencies when investigating this type of uncertainty as substantial under-predictions of LCO amplitude are exhibited.

![Figure 8. Supercritical LCO response, store CG location variability, 700 ft/s](image)

The irregular behaviour shown by the full updating model suggests that some non-physical solutions may be generated due to the lack of LCO at some sample points. Figure 9 shows the effects of the store location at 680 ft/s and 700 ft/s where positive translation of the store describes movement towards the trailing edge. It is apparent that the stability of the wing is very sensitive to the store location, only a small range exhibits LCO, approximately from −0.06 ft to 0.01 ft at 680 ft/s and −0.12 ft to 0.01 ft at 700 ft/s. The wing diverges when only a small movement of the store towards the trailing edge is imposed, the HB model captures this by generating non-physical solutions. When partial updating is used, divergence is predicted at a much later stage than the full updating model. Furthermore, the HB method fails to capture this delayed divergence prediction, a potential danger. Note that the increase of velocity significantly reduces the divergence point for the partial updating case but not the full updating model. As the LCO amplitude increases the agreement between the time domain and HB methods degrades, the LCO behaviour is composed of higher harmonics and it is shown that with the employment of 7 harmonics, the HB method agrees better with the time domain. Both velocities exhibit a sudden reduction in LCO amplitude over a small range for the partial updating case between store locations of 0.2 and 0.3. The reason for this is not fully understood. A feature which is more distinguished in Figure 9(b) is the difference in the manner of the LCO when modeshapes are neglected, the models intersect at the point where the same modeshapes are used (the deterministic case).

Figure 10 shows the effect of the store location variability across a range of velocities. The full updating model produces mostly zero amplitude points and a small amount of the growth of LCOs up to approximately
0.7in before divergence is encountered. By contrast, the partial updating case shows behaviour resembling that of typical supercritical LCO growth. It is worth emphasizing how dangerous partial updating can be for this case due to the over-prediction of the wing stability.

IV.C.2. Subcritical Bifurcations

Figure 11 shows the LCO amplitude variation for the subcritical case at 655ft/s. As in the supercritical case, the full updating provides irregular results with higher amplitudes than the partial updating. Again, a large amount of samples exhibit zero amplitude due to the large changes in stability associated with the position of the store mass. The heat-maps for the subcritical case across different velocities are shown in Figure 12. The full updating model shows similar erratic behaviour as in the supercritical case and the partial updating model shows a noticeable focus of samples on two independent branches, synonymous with subcritical behaviour shown in the wing variability case. The amplitudes predicted using the full updating model are again significantly higher than the results from partial updating. Hence the neglection of the modeshapes is not possible with this type of structural uncertainty.
IV.D. Discussion

Forming conclusions for why the importance of modeshape updating varies so much between the two types of variability requires a deeper look into the behaviour of the wing. One reason for the dependence on modeshapes is the sensitivity of stability due to the uncertainty. The wing’s stability can be assessed by the change in flutter speed inflicted by the variability. Figure 13 shows the distribution flutter speeds with respect to the variability imposed on the model. The wing is much more sensitive to changes in store mass location as there is a larger range of flutter speeds. The wing’s stability increases when the store mass is moved towards the leading edge and decreases when shifted downstream. The greater change in stability is accompanied by greater changes in modeshapes as shown in Figure 14. The modeshapes from the deterministic case are shown in Figure 1(b). Close inspection of the store mass location case shows a shift in elastic axis in all modes to some degree but primarily in mode 2. The modeshapes in the wing case show little change.

The instability of the wing is caused by a hump mode which is described by Beran et al.\textsuperscript{11} Here mode
Figure 13. Variation in flutter speed due to parametric uncertainty, deterministic flutter speed = 658.6 ft/s

Figure 14. Modeshapes for extreme cases

2 is the hump mode which is responsible for flutter when the modal damping becomes positive. Mode 1 causes divergence when it’s damping becomes positive. Instability is also linked with the modal frequencies,
when they become close together or interact the stability of the wing decreases. The modal behaviour of the deterministic case is shown in Figure 15.

![Modal damping and frequencies](image)

(a) Modal damping  
(b) Modal frequencies

Figure 15. Deterministic case mode-tracking

The modal characteristics of the most and least stable samples from the wing variability is in Figure 16 where stability is quantified by flutter speed. The most stable case has a larger starting distance between the frequencies of mode 1 and mode 2, they take longer to interact thus postponing the energy transfer between modes which leads to flutter. For the least stable case the frequencies between mode 1 and mode 2 start interacting at a lower velocity and this causes mode 2 to become positive sooner resulting in a lower flutter speed. Note that for the most stable case mode 4 actually interacts with mode 3 and becomes unstable much earlier, however this occurs after flutter in mode 2 has already taken place.

For the store mass location variability case the differences in modal behaviour are much more pronounced as shown in Figure 17. The interaction between the frequencies from mode 1 and mode 2 are much more prevalent. There is very little interaction for the stable case but the unstable case shows significant coupling between the modes. The most noticeable difference in the damping behaviour is the change in amplitude of the mode 2 hump. This change in amplitude is not seen in the wing variability case. When the mode 2 hump shifts along the velocity axis but does not change amplitude as in the wing case, the variability just linearly shifts the solution along a bifurcation curve which is a function of the variability, this can be captured by the modal frequencies alone. However for the store mass location case, the change in amplitude of the mode 2 damping accompanied with the associated change in flutter speed has a nonlinear relationship with the variability, the modal frequencies cannot capture this. This nonlinear behaviour is manifested in the modeshapes of the structure so they must be included in the analysis for uncertainties in parameters which are significantly affecting individual modes.

V. Conclusions & Outlook

The impact of structural variability on the limit cycle oscillation behaviour of the Goland wing was investigated. Two types of uncertainty were considered:- uncertainty in wing structural parameters such as skin and spar thickness and uncertainty the chord-wise location of the external store’s centre of gravity. The structural variability was imposed in two types of LCO:- supercritical and subcritical. The LCOs were induced by the inclusion of localised polynomial stiffness relationships between the wing tip and the store attachment. Updating the modal frequencies of the model only was compared with updating both the modeshapes and frequencies. For variability in the wing, it was found that the modeshapes were not necessary to achieve accurate computations of the LCO behaviour. This was due to the lack of change in the individual modes imposed by the variability. For the store mass location case, the modeshapes were found to be vital in the simulations as the partial updating could not replicate the actual behaviour of the wing. This was due to the significant change in modal characteristics and stability exhibited by the wing when the position of the store was changed. The consequences of neglecting the modeshapes in this case could be
Figure 16. Wing variability mode-tracking
Figure 17. Store CG location variability, mode-tracking
dangerous as the partial updating model under-predicted LCO amplitudes and over-predicted stability.
Appendix A

The aeroelastic mass, damping and stiffness matrices are given as:

\[
\begin{align*}
\tilde{M}_\phi &= \left[ M_\phi - \frac{\rho V^2}{2} \left( \frac{b}{V} \right)^2 A_2 \right] \\
\tilde{C}_\phi &= \left[ C_\phi - \frac{\rho V^2 b}{2 V} A_1 \right] \\
\tilde{K}_\phi &= \left[ K_\phi - \frac{\rho V^2}{2} A_0 \right]
\end{align*}
\]

The nonlinear restoring force in the spatial domain for the supercritical case is given as:

\[
\{f_{nl}(\dot{q})\} = \begin{bmatrix}
0 \\
\vdots \\
- g_{n1}^2 ([\Phi(q_{1,:})] \{q\} - [\Phi(q_{2,:})] \{q\})^3 \\
- g_{n2}^3 ([\Phi(q_{3,:})] \{q\} - [\Phi(q_{4,:})] \{q\})^3 \\
\vdots \\
g_{n1}^2 ([\Phi(q_{1,:})] \{q\} - [\Phi(q_{2,:})] \{q\})^3 \\
g_{n2}^3 ([\Phi(q_{3,:})] \{q\} - [\Phi(q_{4,:})] \{q\})^3 \\
\vdots \\
0
\end{bmatrix}
\]

Appendix B

The transformation matrices used in eq. (7) are given as:

\[
\begin{align*}
E_{def}^{-1} &= \begin{bmatrix}
[e_{def}]_1 & 0 & \ldots & 0 \\
0 & [e_{def}]_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & [e_{def}]_{n_m}
\end{bmatrix},
[e_{def}] = \begin{bmatrix} 1 \cos \omega t & \ldots & \sin N_H \omega t \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
E_{vel}^{-1} &= \begin{bmatrix}
[e_{vel}]_1 & 0 & \ldots & 0 \\
0 & [e_{vel}]_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & [e_{vel}]_{n_m}
\end{bmatrix},
[e_{vel}] = \begin{bmatrix} 0 & -\omega \sin \omega t & \ldots & \omega N_H \cos N_H \omega t \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
E_{acc}^{-1} &= \begin{bmatrix}
[e_{acc}]_1 & 0 & \ldots & 0 \\
0 & [e_{acc}]_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & [e_{acc}]_{n_m}
\end{bmatrix},
[e_{acc}] = \begin{bmatrix} 0 & -\omega^2 \cos \omega t & \ldots & -(\omega N_H)^2 \sin N_H \omega t \end{bmatrix}
\end{align*}
\]
The convolution integral of the inverse Laplace transform of the generalised aerodynamic matrix yields:

\[ \dot{q}_{a_i} = \int_0^t \dot{q} e^{-\frac{V}{\beta_i}(t-\tau)} d\tau, \quad i = 1, \ldots, n_l \]

Evaluating the integral gives

\[ \dot{q}_{a_i} = \dot{q} \frac{b}{\beta_i} \left( 1 - e^{-\frac{V}{\beta_i} t} \right) \]

Here we are interested in the periodic solution only so the exponential term is damped out which yields:

\[ \dot{q}_{a_i} = \dot{q} \frac{b}{\beta_i}, \quad i = 1, \ldots, n_l \]

The discrete Fourier transforms used in eq. (7) for the \( j \)th mode can then be given as:

\[ \hat{q}^{a,j}_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{b}{\beta_i} \left( \sum_{n=1}^{N_H} (-\omega n \bar{q}^j_{2n-1} \sin \omega n t + \omega n \bar{q}^j_{2n} \cos \omega n t) \right) dt \]

\[ \hat{q}^{a,j}_{2n-1} = \frac{1}{\pi} \int_0^{2\pi} \frac{b}{\beta_i} \left( \sum_{n=1}^{N_H} (-\omega n \bar{q}^j_{2n-1} \sin \omega n t + \omega n \bar{q}^j_{2n} \cos \omega n t) \right) \cos nt \ dt \]

\[ \hat{q}^{a,j}_{2m} = \frac{1}{\pi} \int_0^{2\pi} \frac{b}{\beta_i} \left( \sum_{n=1}^{N_H} (-\omega n \bar{q}^j_{2n-1} \sin \omega n t + \omega n \bar{q}^j_{2n} \cos \omega n t) \right) \sin nt \ dt \]

for \( m = 1, \ldots, N_H \).

The nonlinear force vector can be constructed with Fourier coefficients via a direct substitution of the truncated Fourier series, ie.

\[ \{ f_{nt} \{ \hat{q} \} \} = \begin{bmatrix} 0 \\ \vdots \\ -g^x_{nt}(\{ \Phi(q_1,:) \} \{ \hat{q} \} - [\Phi(q_2,:) \} \{ \hat{q} \})^3 \\ 0 \\ -g^y_{nt}(\{ \Phi(q_3,:) \} \{ \hat{q} \} - [\Phi(q_4,:) \} \{ \hat{q} \})^3 \\ 0 \\ \vdots \\ g^x_{nt}(\{ \Phi(q_1,:) \} \{ \hat{q} \} - [\Phi(q_2,:) \} \{ \hat{q} \})^3 \\ 0 \\ g^y_{nt}(\{ \Phi(q_3,:) \} \{ \hat{q} \} - [\Phi(q_4,:) \} \{ \hat{q} \})^3 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \]

for the supercritical case and
The DFTs for the modal nonlinear force vector in eq. (7) then are:

\[
\{f_{nl}(\tilde{q})\} = \begin{pmatrix}
-h_{nl}^x([\Phi(q_1,:)],[\tilde{q}] - [\Phi(q_2,:)],[\tilde{q}])^5 + g_{nl}^x([\Phi(q_1,:)],[\tilde{q}] - [\Phi(q_2,:)],[\tilde{q}])^3 \\
-h_{nl}^y([\Phi(q_3,:)],[\tilde{q}] - [\Phi(q_4,:)],[\tilde{q}])^5 + g_{nl}^y([\Phi(q_3,:)],[\tilde{q}] - [\Phi(q_4,:)],[\tilde{q}])^3 \\
-h_{nl}^z([\Phi(q_1,:)],[\tilde{q}] - [\Phi(q_2,:)],[\tilde{q}])^5 - g_{nl}^z([\Phi(q_1,:)],[\tilde{q}] - [\Phi(q_2,:)],[\tilde{q}])^3 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix}
\]

for the subcritical case where \(h_{nl}\) is the pentic stiffness coefficient and

\[
\{\tilde{q}\} = \begin{pmatrix}
\tilde{q}_0^1 + \sum_{n=1}^{N_H} (\tilde{q}_{2n-1}^1 \cos \omega nt + \tilde{q}_{2n}^1 \sin \omega nt) \\
\tilde{q}_0^2 + \sum_{n=1}^{N_H} (\tilde{q}_{2n-1}^2 \cos \omega nt + \tilde{q}_{2n}^2 \sin \omega nt) \\
\vdots \\
\tilde{q}_0^m + \sum_{n=1}^{N_H} (\tilde{q}_{2n-1}^m \cos \omega nt + \tilde{q}_{2n}^m \sin \omega nt)
\end{pmatrix}
\]

The DFTs for the modal nonlinear force vector in eq. (7) then are:

\[
\{\tilde{f}_0\} = \frac{1}{2\pi} \int_0^{2\pi} \Phi^T \{f_{nl}(\tilde{q})\} dt
\]

\[
\{\tilde{f}_{2m-1}\} = \frac{1}{\pi} \int_0^{2\pi} \Phi^T \{f_{nl}(\tilde{q})\} \sin mt \, dt
\]

\[
\{\tilde{f}_{2m}\} = \frac{1}{\pi} \int_0^{2\pi} \Phi^T \{f_{nl}(\tilde{q})\} \cos mt \, dt
\]

for \(m = 1, ..., N_H\) and \(\{\tilde{f}_x\} = \begin{bmatrix} \tilde{f}_x^1 & \tilde{f}_x^2 & \ldots & \tilde{f}_x^{N_H} \end{bmatrix}^T\).

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References

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