Nash Equilibria of Network Formation Games under Consent*

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We investigate the Nash equilibria of game theoretic models of network formation based on explicit consent in link formation. These so-called "consent models" explicitly take account of link formation costs. We provide characterizations of Nash equilibria of such consent models under both one-sided and two-sided costs of link formation. We relate these equilibrium concepts to link-based stability concepts, in particular strong link deletion proofness.

Keywords: Social networks, Strong link deletion proofness, Nash equilibrium, Higher orders of stability.

JEL codes: C72, C79, D85

1 Consent in network formation

Networks impact the way we behave, the information we receive, the communities we are part of, and the opportunities we pursue; they determine the machinations of corporations, the benevolence of non-profit organizations, and the workings of the state. Three recent overviews of the work on large scale networks, Watts (1999), Newman (2003) and Newman, Barabasi, and Watts (2006), show the relevance of networks for fields as diverse as physics, social psychology, sociology, and biology. There has been a similar resurgence of interest in economics to understand the phenomenon of network formation. A number of recent contributions to the literature have recognized that networks play an important role in the generation of economic gains by decision makers.

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In this paper we study two game-theoretic models of social network formation.[1] These two models of social network formation are based on *three simple and realistic principles* that govern most real-life networks: (1) Link formation should be based on a binary process of consent; (2) Link formation is in principle costly; and (3) The payoff structure of network formation should be as general as possible.

The process of network formation studied here is a generalization of a simple network formation model developed by Myerson (1991, page 448). Following Myerson, we model the link formation process as a normal form non-cooperative game. This model incorporates the fundamental idea that networks are the result of consensual link formation between pairs of individuals. We augment this model by taking into account the three requirements discussed above and we call this generalization of Myerson's model *the consent model of network formation*.

In our formulation, costs depend on the strategies chosen by the individuals in the link formation process and are incurred independently of the outcome, i.e., even if a link is not established, the initiating individual still has to pay for the act of trying to form that link. In other words, these reflect the cost of "reaching out" to the other individual. We consider both two-sided and one-sided costs of link formation. In the first model, both individuals bear an individually determined cost of link formation, while in the latter model we distinguish between an "initiator" and a "respondent" in the link formation process with only the initiator incurring a link formation cost. This allows us to consider a very general payoff structure that has two components—an arbitrary benefit function and an additive link formation cost structure.[2]

In the literature, the consent model often figures in discussions on network formation but has been portrayed as problematic since it is believed to have "too many" Nash equilibria (Jackson, 2003). However, until now there has been no attempt to provide a complete characterization of the set of these Nash equilibria and our paper tries to address this void in the literature. For both cost structures, we establish the link between the resulting Nash equilibria of the consent model and stable networks founded on well-accepted link-based stability concepts.

For two-sided link formation costs, we establish that a network is supported by a Nash equilibrium if and only if it is strong link deletion proof, in the sense that it is robust against the simultaneous deletion of multiple links with respect to a modified payoff function that explicitly takes into account costs of link formation of only those links that materialize.

Next, we investigate the one-sided cost model where only the link initiating individual incurs a cost. We again devise a modified payoff function that assigns link formation costs to the individual with the lower cost of link formation. If link formation costs are equal, a

[1] Within our framework, we follow standard practice in which the individuals are represented by nodes and their social ties with others by links between these nodes. Nodes and links form together a representation of a social network.

[2] An arbitrary cost structure would require costs to be dependent on outcome. Such a payoff selection would force us to give up the generality of our results. We believe that the chosen payoff structure based on arbitrary benefits and additive link formation costs has the added advantage of capturing what genuinely matters in a realistic process of link formation.

tie-breaking rule is devised. We find that unlike the two-sided cost case, strong link deletion proof networks with respect to this payoff function are supported by Nash equilibria, while the converse does not hold. Also, we address alternative approaches to model one-sided link formation costs, but none result into the desired equivalence.

Finally, we establish relationships between the two cost models under consent in link formation under alternative hypotheses linking the cost structure of the two models. We use the case of uniform network benefits and costs to establish that as one expects, two-sided costs lead to more restrictions on network formation than one-sided link formation costs. Furthermore, we find that for arbitrary configurations, no relationship exists between the Nash equilibria of the two models if the initiator in the model with one-sided costs has to bear both his costs and his partner's costs with regard to the model with two-sided costs. However, if the initiator has to bear only his own costs, then any Nash equilibrium under two-sided link formation costs is also supported by a Nash equilibrium under one-sided link formation costs. The reverse, however, does not hold.

This paper is in many respects complimentary to recent contributions by Hans Haller and his co-authors on the Nash network model (Bala and Goyal, 2000).[3] Haller and co-authors investigated the existence of pure strategy Nash networks in light of the related computational complexity.[4] Haller et al. investigate the relationship between Nash networks introduced by Bala and Goyal (2000) and pairwise stable networks introduced by Jackson and Wolinsky (1996). Another feature that is common to the cited work of Haller et al. and our current paper is the fact that we allow for the value of information generated within the network and the costs of information to be heterogeneous.

Our paper is also closely related to Gilles and Sarangi (2010). There the authors introduce myopic belief systems to overcome the hindrances to link formation identified in the consent approach, resulting into so-called *monadically stable networks*. The focus in that paper is to simply characterize the Nash equilibria of the consent models in terms of established notions of stability in the literature on networks.

The rest of the paper is structured as follows. The next section introduces some notation and terminology. In Section 3 the relation between Nash equilibria of the consent model and link-based stability of networks under two-sided link formation cost of links is discussed. In Section 4, we investigate one-sided link formation cost of links. In Section 5, we compare the two models. Section 6 concludes.

2 Preliminaries

Throughout this paper we consider a given finite set of *individuals* $N = \{1, 2, \ldots, n\}$ with $n \geqslant 2$. In this section, we develop an overview of various well-known concepts from non-cooperative

[3] Nash networks are equilibrium networks in a model of network formation where links can be formed without any requirement for consent.

[4] See Haller and Sarangi (2005) and Haller, Kamphorst, and Sarangi (2007) regarding existence issues and Baron, Durieu, Haller, and Solal (2006) and Baron, Durieu, Haller, Savani, and Solal (2008) regarding the issues pertaining to computational complexity.

game theory and social network theory.

A *non-cooperative game* on the individual set N is given as a list $(A_i, \pi_i)_{i \in N}$ where for every individual $i \in N$, A_i denotes her action set and $\pi_i \colon A \to \mathbb{R}$ is her payoff function, where $A = \prod_{i \in N} A_i$. For every $a \in A$ and $i \in N$, we use $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \in A_{-i} = \prod_{j \neq i} A_j$ to represent the actions selected by the individuals other than i. Throughout we use the abbreviated notation (A, π).

An action $a_i \in A_i$ for individual $i \in N$ is called a *best response* to $a_{-i} \in A_{-i}$ if for every action $b_i \in A_i$ we have that $\pi_i(a_i, a_{-i}) \geqslant \pi_i(b_i, a_{-i})$. An action tuple $a^\star \in A$ is a *Nash equilibrium* of the game (A, π) if for every individual $i \in N$:

$$\pi_i(a^\star) \geqslant \pi_i(b_i, a^\star_{-i}) \text{ for every action } b_i \in A_i.$$

Hence, a Nash equilibrium $a^\star \in A$ satisfies the property that every individual $i \in N$ selects a best response to the actions selected by the other individuals.

2.1 Social networks

Two distinct individuals $i, j \in N$ with $i \neq j$ are said to be *linked* if i and j interact and this interaction results into some socio-economic benefit to both i and j. Such relationships are *undirected* in the sense that both individuals are equal parties in the relationship and neither of them are subjected to authority from the other party. The resulting benefits can be subject to spillover effects, thus allowing for synergies from link formation.

Formally, an (undirected) link between i and j is defined as the set $ij = ji = \{i, j\}$.[5] The collection of all potential links on N is denoted by

$$g_N = \{ij \mid i, j \in N \text{ and } i \neq j\} \tag{1}$$

A *network* g is defined as a collection of links $g \subset g_N$. The collection of all networks on N is denoted by $\mathbb{G}^N = \{g \mid g \subset g_N\}$. The collection $\mathbb{G}^N$ consists of $2^{\frac{1}{2}n(n-1)}$ networks. The network g_N consisting of all links is called the *complete network* on N, and the network $g_0 = \varnothing$ consisting of no links is the *empty network* on N.

For every network $g \in \mathbb{G}^N$ and every individual $i \in N$ we denote i's *neighborhood* in g by

$$N_i(g) = \{j \in N \mid j \neq i \text{ and } ij \in g\} \tag{2}$$

and i's corresponding *direct link set* as

$$L_i(g) = \{ij \mid j \in N_i(g)\} \subset g. \tag{3}$$

The set of all potential links involving i is denoted by $L_i = L_i(g_N) = \{ij \mid j \neq i\}$.

[5] Hence, ij is equivalent to ji, both representing the same undirected relationship between i and j. We delineate link formation costs regarding $\{i, j\}$ by distinguishing the costs c_{ij} incurred by i and the costs c_{ji} incurred by j.

For every pair of individuals $i, j \in N$, we denote by $g + ij$ the network obtained by adding the link $ij \notin g$ to the existing network g, i.e., $g + ij = g \cup \{ij\}$. Also, $g - ij = g \setminus \{ij\}$ denotes the network that results from deleting link $ij \in g$ from the existing network. For any link set $h \subset g$ we denote $g - h = g \setminus h$ and for any link set $h \subset g_N \setminus g$ we define $g + h = g \cup h$.

The payoffs from network formation to the individuals are described by a *network payoff function*, $\varphi \colon \mathbb{G}^N \to \mathbb{R}^N$. It assigns to every individual i a payoff $\varphi_i(g)$ as a function of the network g which include payoffs from direct links as well as spillovers from indirect connections and links between third parties.

2.2 Network-based stability concepts

Next, we introduce different concepts of network stability from a link-based perspective. These concepts rest on the principle that while mutual consent is required for establishing a link, an individual is able to delete her links unilaterally.

(i) A network $g \in \mathbb{G}^N$ is **link deletion proof** if for every individual $i \in N$ and every neighbor $j \in N_i(g)$, it holds that $\varphi_i(g - ij) \leqslant \varphi_i(g)$. Link deletion proofness requires that no individual has an incentive to sever an existing link with one of his neighbors.

(ii) A network $g \in \mathbb{G}^N$ is **strong link deletion proof** (Gilles, Chakrabarti, and Sarangi, 2006) if for every individual $i \in N$ and every set of her direct links $h \subset L_i(g)$, it holds that $\varphi_i(g - h) \leqslant \varphi_i(g)$. Strong link deletion proofness requires that no individual has incentives to sever links with one or more of her neighbors simultaneously.

Obviously, strong link deletion proofness implies link deletion proofness, but not the reverse.

We mention also that Jackson and Wolinsky (1996) introduced the concept of *pairwise stability*, which combines link deletion proofness with the property that, if one individual has a strictly positive gain from adding a certain link to the network, the other party would have strong objections. (The latter property can be denoted as *link addition proofness.*) Pairwise stability will not be used in the present analysis, since myopically strategic behavior only results into link deletion proof or strong link deletion proof networks.

2.3 Myerson's consent game

The consent game was seminally introduced in Myerson (1991, page 448) as an example to illustrate the Nash equilibrium concept. In this simple non-cooperative game, all individuals signal simultaneously their willingness to form links with others. Links are formed if both individuals are agreeable to the link, reflecting the consent requirement. The payoffs are determined fully by the formed network.

Formally, Myerson's consent game is a non-cooperative game (A^m, π^m) on individual set N with for every individual $i \in N$:

$$A_i^m = \{(l_{ij})_{j \neq i} \mid l_{ij} \in \{0, 1\}\}. \tag{4}$$

Here, $l_{ij} = 1$ denotes that individual i wants to form a link with j and $l_{ij} = 0$ signals that individual i does not want to form the link. Obviously, $l_i = (l_{i1}, \dots, l_{i,i-1}, l_{i,i+1}, \dots, l_{in}) \in A_i^m$ denotes a strategy of individual i. A strategy profile is as usual denoted by $l = (l_i)_{i \in N}$.

A link ij is created under consent if both i and j want to form this link, i.e., if $l_{ij} = l_{ji} = 1$. Hence, the resulting network *supported by the strategy profile* $l \in A^m$ is given by

$$g^m(l) = \{ij \in g_N \mid l_{ij} \cdot l_{ji} = 1\}. \tag{5}$$

The consent game is completed by the definition of a payoff function $\pi_i^m \colon A^m \to \mathbb{R}$ with

$$\pi_i^m(l) = \varphi_i\left(g^m(l)\right),$$

where $\varphi \colon \mathbb{G}^N \to \mathbb{R}^N$ is the prevailing network payoff function.

As Myerson (1991) pointed out, for any payoff structure φ, the empty network $g_0 = \varnothing$ is supported through a Nash equilibrium in the Myerson consent game (A^m, π^m).

3 Two-sided link formation costs

In this section, we consider a modification of Myerson's consent game where the "intent to form links" is costly in the sense that approaching another individual to form a link involves *explicit* investment of time, effort and energy. However, if the other individual does not reciprocate and the link does not materialize, the individual choosing to "reach out" still incurs this cost.[6]

Such a cost structure can be represented by a vector $c = (c_{ij})_{j \neq i} \in \mathbb{R}_+^{N \setminus \{i\}}$ where $c_{ij} \geqslant 0$ is the cost that individual i incurs for sending a message to individual j. Here, we assume that an individual always incurs a cost when communicating to another individual. This cost includes responding to messages sent by others. Hence, individual i sending a message $l_{ij} = 1$ to individual j always incurs the cost c_{ij}.

The *consent model with two-sided link formation costs* is defined as a non-cooperative game (A^a, π^a), where individual i's strategy set is given by $A_i^a = A_i^m$ given by (4) and individual i's payoff is given by

$$\pi_i^a(l) = \varphi_i(g^m(l)) - \sum_{j \neq i} l_{ij} \cdot c_{ij} \equiv \pi_i^m(l) - \sum_{j \neq i} l_{ij} \cdot c_{ij}, \tag{6}$$

where $\varphi \colon \mathbb{G}^N \to \mathbb{R}^N$ is the network payoff function representing the gross benefits from network formation without taking into account the costs of link formation.

Our first result develops a complete characterization of the Nash equilibria in the consent model with two-sided link formation costs. Part of this equivalence theorem was already stated without proof in Gilles and Sarangi (2010), but is fully developed here. We introduce a strategy profile that is non-redundant in the construction of the desired social network.

[6]This model of two-sided link formation costs was introduced in Gilles, Chakrabarti, and Sarangi (2006) and developed further by Gilles and Sarangi (2010).

Definition 3.1 *A strategy profile* l *in* (A^a, π^a) *is* ***non-superfluous*** *if for all pairs* i, j, $l_{ij} = 1$ *if and only if* $l_{ji} = 1$.

We call a non-superfluous strategy profile that is a Nash equilibrium a *non-superfluous Nash equilibrium.*

Theorem 1 *Consider an arbitrary network* $g \subset g_N$. *Then the following three statements are equivalent:*

(a) g *is supported by a Nash equilibrium of the consent model of network formation with two-sided link formation costs* (A^a, π^a).

(b) *Network* g *is supported by a non-superfluous Nash equilibrium of the consent model of network formation with two-sided link formation costs* (A^a, π^a).

(c) *Network* g *is strong link deletion proof with regard to the network payoff function* $\varphi^a \colon \mathbb{G}^N \to \mathbb{R}^N$ *given by*

$$\varphi_i^a(g) = \varphi_i(g) - \sum_{j \in N_i(g)} c_{ij} \tag{7}$$

Proof.
(a) implies (c): Let $l^\star$ be an arbitrary Nash equilibrium in (A^a, π^a). Then denote $g^\star = g^m(l^\star) = \{ij \in g_N \mid l^\star_{ij} \cdot l^\star_{ji} = 1\}$. We show that $g^\star$ is strong link deletion proof.
Suppose individual i deletes a certain link set $h_i \subset L_i(g^\star)$. Define $l_i \in A_i^a$ as $l_{ij} = 1$ if $ij \in g^\star - h_i$ and $l_{ij} = 0$ for $ij \notin g^\star - h_i$. Then by $l^\star$ being a Nash equilibrium $g^m(l_i, l^\star_{-i}) = g^\star - h_i$ and $\pi_i^a(l^\star) \geqslant \pi_i^a(l_i, l^\star_{-i})$. Hence,

$$\begin{aligned}
\varphi_i^a(g^\star) &= \varphi_i(g^\star) - \sum_{j \in N_i(g^\star)} c_{ij} = \pi_i^a(l^\star) + \sum_{k\colon l^\star_{ik}=1, l^\star_{ki}=0} c_{ik} \\
&\geqslant \pi_i^a(l^\star) \geqslant \pi_i^a(l_i, l^\star_{-i}) = \varphi_i(g^m(l_i, l^\star_{-i})) - \sum_{k \neq i} l_{ik} \cdot c_{ik} \\
&= \varphi_i(g^\star - h_i) - \sum_{k \in N_i(g^\star - h_i)} c_{ik} = \varphi_i^a(g^\star - h_i).
\end{aligned}$$

This proves that $g^\star$ is strong link deletion proof for φ^a.

(c) implies (b): Suppose that $g^\star \subset g_N$ is a strong link deletion proof network for φ^a. We show that it is supported by a non-superfluous Nash equilibrium strategy in (A^a, π^a). Consider the unique non-superfluous strategy profile $l^\star \in A^a$ such that $g^m(l^\star) = g^\star$. We now show that $l^\star$ is a Nash equilibrium in (A^a, π^a) and $l^\star_{ij} = 1$ if and only if $ij \in g^\star$. Also,

$$\pi_i^a(l^\star) = \varphi_i(g^m(l^\star)) - \sum_{k \neq i} l^\star_{ik} \cdot c_{ik} = \varphi_i(g^\star) - \sum_{k \in N_i(g^\star)} c_{ik} = \varphi_i^a(g^\star).$$

For some individual i consider $l_i \neq l_i^\star$. Define $h_i = \{ik \in g^\star \mid l_{ik} = 0\}$. Then, $g^m(l_i, l^\star_{-i}) = g^\star - h_i$. Since $g^\star$ is strong link deletion proof with respect to φ^a, it follows that $\varphi_i^a(g^\star - h_i) \leqslant$

$\varphi_i^a(g^\star)$. Thus,

$$\begin{aligned}\pi_i^a(l_i, l_{-i}^\star) &= \varphi_i(g^m(l_i, l_{-i}^\star)) - \sum_{k\neq i} l_{ik}\cdot c_{ik} \\ &= \varphi_i(g^\star - h_i) - \sum_{k\in N_i(g^\star - h_i)} c_{ik} - \sum_{k:\, l_{ik}=1, l_{ki}^\star=0} c_{ik} \\ &\leqslant \varphi_i(g^\star - h_i) - \sum_{k\in N_i(g^\star - h_i)} c_{ik} \\ &= \varphi_i^a(g^\star - h_i) \leqslant \varphi_i^a(g^\star) = \pi_i^a(l^\star).\end{aligned}$$

This proves that $l^\star$ is indeed a Nash equilibrium.

Trivially (b) implies (a), which proves the assertion. ■

Theorem 1 provides a complete characterization of the Nash equilibria of the consent model with two-sided link formation costs. As constructed in the proof of Theorem 1, each network is now supported by a unique *non-superfluous strategy profile.*

We mention in passing that there exist superfluous Nash equilibria if costs of link formation are zero for one of the players. We provide a simple example below.

Example 3.2 Consider a network formation situation with $N = \{1, 2\}$ and $\varphi_1(g_0) = \varphi_2(g_0) = \varphi_1(g_N) = 0$ and $\varphi_2(g_N) = 1$. Link formation costs are given by $c_{12} = 0$ and $c_{21} = 1$. Hence for $i = 1, 2$, $\varphi_i^a(g_0) = 0$ as well as $\varphi_i^a(g_N) = 0$. Clearly, the empty network g_0 is both (strong) link deletion proof for the net payoff function φ^a and supported by the superfluous Nash equilibrium $l_{12} = 1$; $l_{21} = 0$. Of course, g_0 is also supported as a Nash equilibrium through its non-superfluous strategy profile in (A^a, π^a). ♦

4 One-sided link formation costs

Next we investigate the properties of equilibria in a network formation process under a one-sided cost structure. In this approach, one of the two individuals acts as the *initiator* and sends an initiation message to the other. If the other individual, called the *responder*, chooses to reciprocate positively, the link materializes; otherwise, not. This link formation process is similar to the one considered in Bala and Goyal (2000), except that in our approach the responder has to consent to the formation of the link, while in Bala-Goyal's model, this is not required.

The decision making process is more complex than that under two-sided link formation costs. Consequently, the action set is different and, for each individual i, is given by

$$A_i^b = \left\{ (l_{ij}, r_{ij})_{j\neq i} \;\middle|\; l_{ij}, r_{ij} \in \{0, 1\} \right\}. \tag{8}$$

Player i chooses to act as an initiator in forming a link with j if she initiates a message to j indicated as $l_{ij} = 1$. Player j responds positively to this initiative if $r_{ji} = 1$, and player j

rejects the initiated link if $r_{ji} = 0$. Therefore, a link is only established if the initiated link is accepted, i.e., if $l_{ij} = r_{ji} = 1$. This is formalized as follows.

Let $A^b = \prod_{i \in N} A_i^b$. Given the described link formation process, for any $(l, r) \in A^b$, the resulting network is now given by

$$g^b(l, r) = \{ij \in g_N \mid l_{ij} = r_{ji} = 1\}. \tag{9}$$

When individual i initiates a link with individual j—represented by $l_{ij} = 1$—she incurs a cost of $\gamma_{ij} \geqslant 0$, regardless of whether the initialized link is accepted by j or not. Responding to a link initialization message on the other hand is costless. This results in the following net payoff function for individual i:

$$\pi_i^b(l, r) = \varphi_i(g^b(l, r)) - \sum_{j \neq i} l_{ij} \cdot \gamma_{ij}. \tag{10}$$

We refer to the game (A^b, π^b) as the *consent model of network formation with one-sided link formation costs.* We can construct a non-superfluous strategy profile as follows:

Definition 4.1 *A strategy profile* (l, r) *of* (A^b, π^b) *is* ***non-superfluous*** *if for all pairs* $i, j \in N$ *it holds that*

$$l_{ij} = 1 \textit{ implies that } r_{ji} = 1 \textit{ as well as } l_{ji} = r_{ij} = 0, \textit{ and} \tag{11}$$

$$r_{ij} = 1 \textit{ implies that } l_{ji} = 1 \textit{ as well as } l_{ij} = r_{ji} = 0. \tag{12}$$

Unlike for the model under two-sided link formation costs, each network is no longer supported by an unique non-superfluous strategy profile. But under a non-superfluous strategy profile, only one individual bears the establishment cost of each existing link, and every initialization is responded to positively. Next, we consider the relationship between the Nash equilibria of the two-sided and the one-sided model and characterization of the Nash equilibria of the one-sided model in terms of stability properties.

We first address whether there is a network payoff function which would provide equivalence between Nash equilibria of the one-sided model and stability with regard to a payoff function in a similar vein as Theorem 1 for two-sided link formation costs. In particular, we construct a payoff function which only assigns link formation costs to the individual with the lower cost of link formation. If link formation costs are equal, a tie-breaking rule is applied.

Let $\Omega_i(g) = \{j \in N_i(g) \mid \gamma_{ij} < \gamma_{ji}$ or $\gamma_{ij} = \gamma_{ji}$, $i < j\} \subset N_i(g)$. Now the payoff function φ^b is defined for $i \in N$ by

$$\varphi_i^b(g) = \varphi_i(g) - \sum_{j \in \Omega_i(g)} \gamma_{ij}$$

given any network payoff function φ representing benefits without taking into account costs of link formation. We can show the following implication.

Theorem 2 *If a network* $g^\star \subset g_N$ *is strong link deletion proof for the net payoff function* φ^b*, then it is supported by a non-superfluous Nash equilibrium of the consent model with one-sided link formation costs.*

Proof. Let $g^\star$ be strong link deletion proof under the net payoff function φ^b. For $g^\star$, define a non-superfluous strategy tuple $\lambda^\star = (l^\star, r^\star)$ as follows: $l^\star_{ij} = r^\star_{ji} = 1$ if $ij \in g$ and $\gamma_{ij} < \gamma_{ji}$ or $\gamma_{ij} = \gamma_{ji}$, $i < j$; and $l^\star_{ij} = r^\star_{ji} = 0$ otherwise. Obviously, $g^b(l^\star, r^\star) = g^\star$ and

$$\pi^b_i(\lambda^\star) = \varphi_i(g^b(\lambda^\star)) - \sum_{j \neq i} l^\star_{ij} \cdot \gamma_{ij} = \varphi_i(g^\star) - \sum_{j \in \Omega_i(g^\star)} \gamma_{ij} = \varphi^b_i(g^\star).$$

Now, for individual i consider an arbitrary deviation $\widehat{\lambda}_i = (\widehat{l}_i, \widehat{r}_i) \neq (l^\star_i, r^\star_i) = \lambda^\star_i$. In any such deviation, no new links will be formed because if $ij \notin g^\star$, it follows that $l^\star_{ji} = r^\star_{ji} = 0$. However, links in i's neighborhood link set can be deleted. Hence let $g^b(\widehat{\lambda}_i, \lambda^\star_{-i}) = g^\star - h_i$ where $h_i \subset L_i(g^\star)$.

We now prove that $j \in N_i(g^\star - h_i)$ and $\left[\gamma_{ij} < \gamma_{ji} \text{ or } \gamma_{ij} = \gamma_{ji},\ i < j\right]$ implies that $\widehat{l}_{ij} = 1$. In other words, $j \in \Omega_i(g^\star - h_i) \subset N_i(g^\star - h_i)$ implies that $\widehat{l}_{ij} = 1$.

To obtain a contradiction, let $j \in \Omega_i(g^\star - h_i)$ but $\widehat{l}_{ij} = 0$. Now,

$$j \in N_i(g^\star - h_i) \Leftrightarrow \widehat{l}_{ij} = 1 \text{ and } r^\star_{ji} = 1 \text{ or } \widehat{r}_{ij} = 1 \text{ and } l^\star_{ji} = 1. \tag{13}$$

But $l^\star_{ji} = 1$ implies by construction $\gamma_{ij} > \gamma_{ji}$ or $\gamma_{ij} = \gamma_{ji}, i > j$ and $r^\star_{ji} = 1$ implies by construction that $\gamma_{ij} < \gamma_{ji}$ or $\gamma_{ij} = \gamma_{ji}$, $i < j$. Since $\widehat{l}_{ij} = 0$, by (13), it follows that $\widehat{r}_{ij} = l^\star_{ji} = 1$ which implies that $\gamma_{ij} > \gamma_{ji}$ or $\gamma_{ij} = \gamma_{ji}$ with $i > j$. This contradicts $j \in \Omega_i(g^\star - h_i)$ completing the proof of the claim stated above.

Now, the proven claim implies

$$\sum_{j \in \Omega_i(g^\star - h_i)} \gamma_{ij} \leqslant \sum_{j \in N_i(g^\star - h_i)} \widehat{l}_{ij} \cdot \gamma_{ij} \leqslant \sum_{j \neq i} \widehat{l}_{ij} \cdot \gamma_{ij}. \tag{14}$$

Hence,

$$\begin{aligned}
\pi^b_i(\widehat{\lambda}_i, \lambda^\star_{-i}) &= \varphi_i(g^b(\widehat{\lambda}_i, \lambda^\star_{-i})) - \sum_{j \neq i} \widehat{l}_{ij} \cdot \gamma_{ij} = \varphi_i(g^\star - h_i) - \sum_{j \neq i} \widehat{l}_{ij} \cdot \gamma_{ij} \\
&\leqslant \varphi_i(g^\star - h_i) - \sum_{j \in \Omega_i(g^\star - h_i)} \gamma_{ij} = \varphi^b_i(g^\star - h_i) \\
&\leqslant \varphi^b_i(g^\star) = \pi^b_i(l^\star, r^\star).
\end{aligned}$$

The first inequality follows from (14) and the second follows from the fact that $g^\star$ is strong link deletion proof with respect to φ^b. ∎

The converse of Theorem 2 does not hold as shown by the following counter-example.

Example 4.2 Consider a network formation situation with $N = \{1, 2\}$ and $\varphi_1(g_0) = \varphi_2(g_0) = 0$, $\varphi_1(g_N) = 2$ and $\varphi_2(g_N) = 10$. Link formation costs are given by $\gamma_{12} = 5$ and $\gamma_{21} = 7$.

Hence for $i = 1, 2$, $\varphi_i^b(g_0) = 0$ and $\varphi_1^b(g_N) = -3$ and $\varphi_2^b(g_N) = 10$. Clearly, the complete network is not (strong) link deletion proof for the net payoff function φ^b. But there is a Nash equilibrium of the one-sided model that supports it: $l_{12} = 0$; $r_{12} = 1$; $l_{21} = 1$; $r_{21} = 0$. ♦

The problem of finding a reasonable payoff function that can completely characterize Nash equilibria of the one-sided model in terms of stability remains open. One might expect that a network payoff function that assigns a link initiator role to the individual with the higher marginal net benefits as a result of formation of the link in question might resolve the issue. Below it is shown that this is not true. In fact, we conjecture that no such payoff function exists.

Example 4.3 Consider a situation with three players, 1, 2 and 3. The following table gives the benefits for each of the three individuals in the case of the formation of one of three relevant networks:

Network g	$\varphi_1(g)$	$\varphi_2(g)$	$\varphi_3(g)$
$\{12\}$	10	10	0
$\{13\}$	10	0	10
$\{12, 13\}$	15	20	20

All other networks generate no benefits to any of the three individuals, i.e., $\varphi_i(g) = 0$ for all other networks g not listed in the table.
Consider the following link formation costs: $\gamma_{12} = \gamma_{13} = 9$, $\gamma_{21} = 10$, $\gamma_{31} = 10$, and $\gamma_{23} = \gamma_{32} = 10$. Within this context, individual 1 has the highest marginal net benefit from forming links 12 as well as 13, namely $\varphi_1(\{12\}) - \gamma_{12} = \varphi_1(\{13\}) - \gamma_{13} = 1$, while the other players have no net benefits from forming links 12 and 13.
Now, the network $\{12, 13\}$is not link deletion proof if the individual with the highest net marginal benefit is assumed to finance the formation of a link. Indeed, individual 1—who has the highest net marginal benefits from both links—has a negative net benefit from $\{12, 13\}$ and would prefer to sever one of the two links to increase her net benefit to 1.
On the other hand, $\{12, 13\}$ is supported by a non-superfluous Nash equilibrium strategy profile under one-sided link formation costs with $l_{21} = r_{12} = 1$ and $l_{31} = r_{13} = 1$. ♦

Next we examine whether certain other refinements can resolve the coordination and free riding issues and restore equivalence between Nash equilibria of the model with one-sided costs and strong link deletion proofness with respect to some well-constructed network payoff function.

Often sequential decision making solves a coordination problem. With this in mind, consider the following two-stage game: In the first stage, every individuals $i \in N$ initiates links by determining $(l_{ij})_{j \neq i}$. In the second stage, all individuals respond to links initiated in the first stage and select $(r_{ij})_{j \neq i}$. The question is whether the subgame perfect Nash equilibria of this game are strong link deletion proof with regard to φ^b. We show that this is not necessarily the case.

Example 4.4 Reconsider Example 4.2. We showed earlier that the complete network is not (strong) link deletion proof for the net payoff function φ^b but there is a Nash equilibrium of the one-sided model that supports it, namely, $l_{12} = 0$; $r_{12} = 1$; $l_{21} = 1$; $r_{21} = 0$. We now show that in the new game, this is subgame perfect as well. Consider the reduced game in the second stage, given that $l_{12} = 0$ and $l_{21} = 1$ has been chosen in the first stage. In normal form it can now be represented as

r_{12} \ r_{21}	0	1
0	$0, -7$	$0, -7$
1	$2, 3$	$2, 3$

There are two Nash equilibria in this game, one of which is $r_{12} = 1$ and $r_{21} = 0$. It is easy to check that the given strategy tuple is indeed a subgame perfect equilibrium in this game. ♦

The reason why sequential decision making cannot resolve the coordination problem is that here the problem stems from costs not being transferable. Complete transferability of costs and benefits would take us into the framework of Jackson and Wolinsky (1996) and Bloch and Jackson (2007).

5 A comparison of the two cost models

In this section we compare the two cost models of network formation that we have introduced in the previous sections. We first develop an elaborate application that illustrates the differences that emerge in under the two cost models of link formation. Second, we provide a formal comparison of the Nash equilibria under two-sided and one-sided link formation costs.

5.1 Uniform network benefits

Consider a scenario where the gross benefit function $\varphi_i : \mathbb{G}^N \to \mathbb{R}$ is uniform across all the individuals in any component[7] and fully described by $\varphi_i(g) = \omega(k)$, where $\omega \colon \mathbb{N} \to \mathbb{R}_+$ is a strictly increasing and strictly concave function in the size k of the component of which individual i is a member. A well-known example of a gross benefit function that satisfies these requirements is the payoff function stemming from trade networks originally due to Jackson and Watts (2002, page 274) and subsequently elaborated upon in Jackson and van den Nouweland (2005, page 428). We further assume that all link formation costs are uniform across all individuals, given by $c \geqslant 0$ for the two-sided model and $\gamma \geqslant 0$ for the one-sided model.

We first investigate the two-sided link formation cost model. Since benefits are uniform across individuals, the stability of the network depends on the individual who has the least net payoff stemming from maintaining a certain link, denoted as the *consequential* individual.

[7]We recall that two individuals are *connected* if there exists a path between these two individuals in the prevailing network. Now, a *component* in a given network is a maximally connected group of individuals.

For two-sided link formation costs, this is exactly the individual who has the least to lose in terms of gross benefits by deleting a certain link. For a star network, this individual is given by the center of the star.

Define the function ρ as the marginal benefit function given by

$$\rho(k) = \omega(k) - \omega(k-1). \tag{15}$$

Given that ω is strictly concave, ρ is strictly decreasing.

The consequential individual is interested in maintaining a link, if $\rho(k) \geqslant c$. In other words, the largest star component in a network that can be supported by a Nash equilibrium of the consent model with two-sided link formation costs is given by $\tilde{k}$ with $\rho(\tilde{k}) \geqslant c$ and $\rho(\tilde{k}+1) < c$.

For the one-sided link formation cost model, the consequential individual is no longer that individual who has the least to lose by deleting a link—because that individual can act as the responder rather than the initiator—but the individual in the neighborhood of this individual, who has the least to lose by deleting the link with the least net benefit individual is the consequential individual. Consequently, the largest star network that can be supported under one-sided link formation costs is equal to n if $\gamma \leqslant \omega(n)$ and zero otherwise.

Assuming that the two cost parameters c and γ are not too different from each other, we emphasize the contrasting effects on the star network architectures as we move from two-sided to one sided link formation costs. It is not surprising that larger (connected) networks can be supported under one-sided link formation costs, but the extremely large increase of the supported size from $\tilde{k}$ to n is rather surprising.

5.2 A formal relationship between the two consent models

Since the models have different philosophical bases, we must make some simplifying assumptions to enable a more formal comparison.

Case A: Suppose that the initiator in the model with one-sided costs bears both his cost and the cost of the responder in context of the two sided model. So, initiation is tantamount to bearing the total cost of link formation, i.e., $\gamma_{ij} = c_{ij} + c_{ji}$ for all $i \neq j$. Benefits remain individualized and are not transferable.

In this case, it is quite obvious that the Nash equilibria of the two models are not comparable:

Example 5.1 Let $N = \{1,2\}$ and $\varphi_i(g_N) = 51$, $\varphi_i(g_0) = 0$, $i = 1,2$ and $c_{12} = c_{21} = 50$. Then, g_N is supported by a Nash equilibria of the two sided model, namely $l_{12} = l_{21} = 1$. But there is no Nash equilibrium in the one-sided model that would support it because no one would be willing to sustain a cost of 100 in order to sustain this network.
Next, let $\varphi_1(g_N) = 12$, $\varphi_2(g_N) = 2$, $\varphi_i(g_0) = 0$, $i = 1,2$ and $c_{12} = c_{21} = 5$. Then, g_N is supported by a Nash equilibria of the one sided model, namely $l_{12} = r_{21} = 1$, $l_{21} = r_{12} = 0$. The strategy supporting this network in the two-sided model is not a Nash equilibrium. ♦

Case B: Next, we consider the case in which the link formation costs are not transferable and that the initiator has to bear only his own cost.[8] Namely, $\gamma_{ij} = c_{ij}$ for all $i \neq j$. In this case, it can be shown that networks supported by Nash equilibria of the two two-sided model are also supported by some Nash equilibrium of the one-sided model, while the converse does not hold.

Theorem 3 *If a network is supported by a Nash equilibrium of the consent model with two-sided link formation costs, then there exists a non-superfluous Nash equilibrium in the consent model with one-sided link formation costs supporting this network.*

In order to prove Theorem 3 we require a preliminary result.

Lemma 5.2 *Consider the network* g *supported by a non-superfluous strategy profile* (l, r). *Consider a deviation by an individual* i *to* $(\tilde{l}_i, \tilde{r}_i) \neq (l_i, r_i)$ *yielding a network* $\tilde{g}$. *Then* $\tilde{g} \subset g$.

Proof. The only fact that needs to be shown is that if $ij \notin g$, $j \neq i$, it does not belong to $\tilde{g}$ either. Now, $ij \notin g \Leftrightarrow l_{ij} = r_{ji} = l_{ji} = r_{ij} = 0$ by definition. But $l_{ji} = r_{ji} = 0$ implies $ij \notin \tilde{g}$. This proves the claim. ■

Proof of Theorem 3. Let $g^\star$ be supported by a Nash equilibrium $l^\star \in A^a$ of the consent model with two-sided link formation costs (A^a, π^a). We now construct a non-superfluous strategy tuple $(\hat{l}, \hat{r}) \in A^b$ in the consent model with one-sided link formation costs such that $g^b(\hat{l}, \hat{r}) = g^\star$ and $(\hat{l}, \hat{r})$ is a Nash equilibrium in (A^b, π^b).

From Theorem 1, we can assume without loss of generality that $l^\star \in A^a$ is non-superfluous. Given $l^\star$, we define $\widehat{\lambda} = (\widehat{l}, \widehat{r}) \in A^b$ by

(a) $\widehat{l}_{ij} = \widehat{r}_{ji} = 1$ and $\widehat{l}_{ji} = \widehat{r}_{ij} = 0$ if and only if $l^\star_{ij} = l^\star_{ji} = 1$, and either $c_{ij} < c_{ji}$, or $c_{ij} = c_{ji}$ and $i < j$.

(b) $\widehat{l}_{ij} = \widehat{r}_{ji} = 0$ and $\widehat{l}_{ji} = \widehat{r}_{ij} = 1$ if and only if $l^\star_{ij} = l^\star_{ji} = 1$, and either $c_{ij} > c_{ji}$, or $c_{ij} = c_{ji}$ and $i > j$.

(c) $\widehat{l}_{ij} = \widehat{l}_{ji} = \widehat{r}_{ij} = \widehat{r}_{ji} = 0$ if and only if $l^\star_{ij} = l^\star_{ji} = 0$.

It follows immediately that $(\widehat{l}, \widehat{r})$ is a non-superfluous strategy profile supporting $g^b(\widehat{l}, \widehat{r}) = g^\star$. It remains to be shown that $(\widehat{l}, \widehat{r})$ is a Nash equilibrium of the consent model with one sided link formation costs.

Let an arbitrary individual i select strategies $(\tilde{l}_i, \tilde{r}_i) \in A^b_i$. Note that $\widehat{\lambda}_i = (\widehat{l}_i, \widehat{r}_i)$ and define $\overline{\lambda} = (\tilde{\lambda}_i, \widehat{\lambda}_{-i})$, which is the resulting strategy tuple following deviation by i.

Now let $\Lambda_{i1}(\overline{\lambda}) = \{j \in N \mid \tilde{l}_{ij} = 0, \tilde{r}_{ij} = 1, \widehat{l}_{ji} = 1\}$ and $\Lambda_{i0}(\overline{\lambda}) = \{j \in N \mid \tilde{l}_{ij} = 1, \widehat{r}_{ji} = 0, \min(\widehat{l}_{ji}, \tilde{r}_{ij}) = 0\}$. We know from Lemma 5.2 that $g^b(\overline{\lambda}) \subset g^\star$. Hence, given that $l^\star$ is a Nash equilibrium and, therefore, by Theorem 1, $g^\star$ is strong link deletion proof with respect to φ^a, implying that

$$\varphi^a_i(g^b(\overline{\lambda})) \leqslant \varphi^a_i(g^\star) \tag{16}$$

[8]One can think of a scenario where the costs of one party are sunk and thus not relevant to decision making.

Now,

$$\pi_i^b(\bar\lambda) = \varphi_i(g^b(\bar\lambda)) - \sum_{ij\in g^b(\bar\lambda)} c_{ij} + \sum_{j\in\Lambda_{i1}(\bar\lambda)} c_{ij} - \sum_{j\in\Lambda_{i0}(\bar\lambda)} c_{ij}.$$

We assign to individual i the costs of all the links in $g^b(\bar\lambda)$ and then deduct those costs for which i acts as the receptor in the new strategy profile $\bar\lambda$ and hence are not borne by individual i. Hence,

$$\begin{aligned}
\pi_i^b(\bar\lambda) &\leqslant \varphi_i(g^b(\bar\lambda)) - \sum_{ij\in g^b(\bar\lambda)} c_{ij} + \sum_{j\in\Lambda_{i1}(\bar\lambda)} c_{ij} \\
&= \varphi_i^a(g^b(\bar\lambda)) + \sum_{j\neq i} c_{ij}\cdot\tilde r_{ij}\cdot\hat r_{ij}\cdot\left(1-\tilde l_{ij}\right) && (17)\\
&\leqslant \varphi_i^a(g^\star) + \sum_{j\neq i} c_{ij}\cdot\tilde r_{ij}\cdot\hat r_{ij}\cdot\left(1-\tilde l_{ij}\right) && (18)\\
&= \varphi_i(g^\star) - \sum_{ij\in g^\star} c_{ij} + \sum_{j\neq i} c_{ij}\cdot\tilde r_{ij}\cdot\hat r_{ij}\cdot\left(1-\tilde l_{ij}\right)\\
&\leqslant \varphi_i(g^\star) - \sum_{ij\in g^\star} c_{ij} + \sum_{j\neq i} c_{ij}\cdot\tilde r_{ij}\cdot\hat r_{ij} && (19)\\
&= \varphi_i(g^\star) - \sum_{j\neq i}\hat l_{ij}\cdot c_{ij} - \sum_{j\neq i}\hat r_{ij}\cdot c_{ij} + \sum_{j\neq i}\tilde r_{ij}\cdot\hat r_{ij}\cdot c_{ij}\\
&= \pi_i^b(\hat l,\hat r) - \left[\sum_{j\neq i}\hat r_{ij}\cdot c_{ij} - \sum_{j\neq i}\tilde r_{ij}\cdot\hat r_{ij}\cdot c_{ij}\right] \leqslant \pi_i^b(\hat l,\hat r). && (20)
\end{aligned}$$

To show the derivation above, we consider the various inequalities:
Equality (17) holds because $j\in\Lambda_{i1}(\bar\lambda)$ implies $\tilde l_{ij}=0, \tilde r_{ij}=1, \hat l_{ji}=1$. This implies $(1-\tilde l_{ij})=1$ and $\tilde r_{ij}=1$. Furthermore, by definition $\hat l_{ji}=1$ implies $\hat r_{ij}=1$. If $j\notin\Lambda_{i1}(\bar\lambda)$, either $\tilde l_{ij}=1$ or $\tilde r_{ij}=0$ or $\hat l_{ji}=0$. Either of these reduce the expression to zero noting the fact that $\hat l_{ji}=0$ implies $\hat r_{ij}=0$.
The second inequality (18) follows from (16). The third inequality (19) is again obvious given $(1-\tilde l_{ij})\leqslant 1$. The fourth inequality (20) follows from the fact that $\tilde r_{ij}\leqslant 1$.
This completes the proof of the assertion. ■

We show that the converse of Theorem 3 does not hold.

Example 5.3 Consider a situation with two individuals, $N=\{1,2\}$ with $\varphi_1(g_0)=\varphi_2(g_0)=0$, $\varphi_1(g_N)=6$ and $\varphi_2(g_N)=4$. Let costs of link formation be 5 for each individual. The complete network initiated by individual 1 is supported by a Nash equilibrium in the one-sided model. But the strategy profile in the two-sided model that supports this network is not a Nash equilibrium. ♦

6 Concluding remarks

In this paper we have investigated network formation under mutual consent and costly communication. We developed two modifications of Myerson's seminal model of network formation under mutual consent that reflect such link formation cost considerations.

We believe that the two approaches to bearing link formation costs under consent opens the way to investigate link formation processes in more precise detail. This requires the introduction of a dynamic approach to the link formation process, as already explored for a two-stage process of initiation and response in Example 4.3. This example makes clear that a deeper understanding is required that takes us to more advanced models such as captured by *farsightedness* in network formation (Page, Wooders, and Kamat, 2002; Dutta, Ghosal, and Ray, 2005; Page, Wooders, and Kamat, 2005; Herings, Mauleon, and Vannetelbosch, 2009). It should be expected that in such dynamic approaches complete equivalence results will be hard to establish and more direct methodologies are called for.

References

Bala, V., and S. Goyal (2000): "A Non-Cooperative Model of Network Formation," *Econometrica*, 68, 1181–1230.

Baron, R., J. Durieu, H. Haller, R. Savani, and P. Solal (2008): "Good Neighbors are Hard to Find: Computational Complexity of Network Formation," *Review of Economic Design*, 12, 1–19.

Baron, R., J. Durieu, H. Haller, and P. Solal (2006): "Complexity and Stochastic Evolution of Dyadic Networks," *Computers & Operations Research*, 33, 312–327.

Bloch, F., and M. O. Jackson (2007): "The Formation of Networks with Transfers among Players," *Journal of Economic Theory*, 133, 83–110.

Dutta, B., S. Ghosal, and D. Ray (2005): "Farsighted network formation," *Journal of Economic Theory*, 122(2), 143–164.

Gilles, R. P., S. Chakrabarti, and S. Sarangi (2006): "Social Network Formation with Consent: Nash Equilibrium and Pairwise Refinements," Working paper, Department of Economics, Virginia Tech, Blacksburg, VA.

Gilles, R. P., and S. Sarangi (2010): "Network Formation under Mutual Consent and Costly Communication," *Mathematical Social Sciences*, 60(3), 181–185.

Haller, H., J. Kamphorst, and S. Sarangi (2007): "(Non-) Existence and Scope of Nash Networks," *Economic Theory*, 31, 597–604.

Haller, H., and S. Sarangi (2005): "Nash Networks with Heterogeneous Agents," *Mathematical Social Sciences*, 50, 181–201.

Herings, P. J.-J., A. Mauleon, and V. Vannetelbosch (2009): "Farsightedly stable networks," *Games and Economic Behavior*, 67, 526–541.

Jackson, M. O. (2003): “The Stability and Efficiency of Economic and Social Networks,” in *Networks and Groups: Models of Strategic Formation*, ed. by B. Dutta, and M. Jackson. Springer Verlag, New York, NY.

Jackson, M. O., and A. van den Nouweland (2005): “Strongly Stable Networks,” *Games and Economic Behavior*, 51, 420–444.

Jackson, M. O., and A. Watts (2002): “The Existence of Pairwise Stable Networks,” *Seoul Journal of Economics*, 14(3), 299–321.

Jackson, M. O., and A. Wolinsky (1996): “A Strategic Model of Social and Economic Networks,” *Journal of Economic Theory*, 71, 44–74.

Myerson, R. B. (1991): *Game Theory: Analysis of Conflict*. Harvard University Press, Cambridge, MA.

Newman, M. (2003): “The structure and function of complex networks,” *SIAM Review*, 45, 167–256.

Newman, M., A.-L. Barabasi, and D. Watts (2006): *The Structure and Dynamics of Networks*, Princeton Series in Complexity. Princeton UP, Princeton, NJ.

Page, F. H., M. H. Wooders, and S. Kamat (2002): “Network and Farsighted Stability,” Working Paper, University of Warwick, Warwick, UK.

——— (2005): “Networks and farsighted stability,” *Journal of Economic Theory*, 120, 257—269.

Watts, D. J. (1999): *Small Worlds: The Dynamics of Networks between Order and Randomness*, Princeton Studies in Complexity. Princeton UP, Princeton, NJ.